On Fundamental Groups of Galois Closures of Generic Projections

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Contents

Introduction

Schon winkt der Wein im gold'nen Pokale,
Doch trinkt noch nicht, erst sing' ich euch ein Lied!
Das Lied vom Kummer
Soll auflachend in die Seele euch klingen.

Fundamental groups are birational invariants of smooth algebraic varieties and for a classification it is important to know them. Also it is interesting to see how much of the classification is encoded in them.

These groups are known for smooth and complex quasi-projective curves. For smooth and complex projective curves the fundamental group determines the curve up to deformation of the complex structure. For surfaces the situation is much more complicated. The classification of surfaces is still not complete and mainly surfaces of general type are still not well understood. In particular, surfaces of general type with $K^2 \geq 8\chi$ seemed to be mysterious and were hard to construct. Bogomolov and others conjectured that these surfaces have infinite fundamental groups.

Miyaoka considered generic projections from smooth projective surfaces to the projective plane and studied the Galois closures of these projections. He was able to construct many surfaces of general type with $K^2 \geq 8\chi$ via this method.

Moishezon and Teicher showed that there are generic projections from $\mathbb{P}^1 \times \mathbb{P}^1$ such that the corresponding Galois closures are simply connected and fulfill $K^2 \geq 8\chi$. These were the first counter-examples to the conjecture mentioned above. Their proof involved a certain amount of computations and was based on degeneration techniques and braid monodromy factorisations.

In this thesis we attack the problem of determining the fundamental group of the Galois closure of a generic projection via determining some "obvious" contributions coming from X. So let $f : X \to \mathbb{P}^2$ be a generic projection of degree n and let X_{gal} be the corresponding Galois closure. It is known that X_{gal} embeds into X^n which induces a homomorphism of fundamental groups

$$\pi_1(X_{\text{gal}}) \quad \to \quad \pi_1(X)^n \,. \tag{1}$$

If we denote by $\mathcal{K}(G, n)$ the kernel of the homomorphism from G^n onto G^{ab} then the image of (1) is precisely $\mathcal{K}(\pi_1(X), n)$. We prove this by purely algebraic methods. In particular, we obtain this result also for étale fundamental groups and generic projections defined over algebraically closed fields of characteristic $\neq 2, 3$.

Over the complex numbers there is the algorithm of Zariski and van Kampen to determine the fundamental group of the complement of a curve in the affine or projective plane. Since the monodromy at infinity is a little bit tricky, it is easier to look at the affine situation first, i.e. to look at the fibre of f and f_{gal} over a generically chosen affine plane in $\mathbb{P}^2$. We refer to these fibres as X^{aff} and $X^{\text{aff}}_{\text{gal}}$, respectively.

In the affine situation there is also a surjective homomorphism from $\pi_1(X^{\text{aff}}_{\text{gal}})$ onto $\mathcal{K}(\pi_1(X^{\text{aff}}), n)$ as in (1). Using the algorithm of Zariski and van Kampen we obtain a quotient

$$\pi_1(X^{\text{aff}}_{\text{gal}}) \quad \twoheadrightarrow \quad \widetilde{\mathcal{K}}(\pi_1(X^{\text{aff}}),\, n). \tag{2}$$

Here, $\widetilde{\mathcal{K}}(G, n)$ is a purely group theoretical construction that can be defined for every finitely generated group G and every natural number $n \geq 3$. It is related to $\mathcal{K}(G, n)$ via a short exact and central sequence

$$0 \quad \to \quad H_2(G,\, \mathbb{Z}) \quad \to \quad \widetilde{\mathcal{K}}(G, n) \quad \to \quad \mathcal{K}(G, n) \quad \to \quad 1 \tag{3}$$

where $H_2(G, \mathbb{Z})$ denotes the second group homology with integral coefficients. Even though the computation of $\mathcal{K}(G, n)$ for a given group G is usually not so complicated it is quite hard to say something about $\widetilde{\mathcal{K}}(G, n)$ and therefore about about the quotient (2) of $\pi_1(X^{\text{aff}}_{\text{gal}})$ in general.

Also we deduce from (3) that the quotient of $\pi_1(X^{\text{aff}}_{\text{gal}})$ computed by (2) is usually larger than the one given by (1).

It remains to determine the kernel of the homomorphism (2). We show that it is a naturally defined subgroup that can be formulated independent of the specific situation. We denote by $R_{\text{gal}} \subset X_{\text{gal}}$ the ramification locus of f_{gal}. This divisor is ample but it is not irreducible. Then the kernel of (2) is trivial if the inverse image of R_{gal} in the universal cover of $X^{\text{aff}}_{\text{gal}}$ has certain connectivity properties. Thus if these hold true then $\pi_1(X^{\text{aff}}_{\text{gal}})$ is isomorphic to $\widetilde{\mathcal{K}}(\pi_1(X^{\text{aff}}), n)$.

It is interesting to see that in all known examples (except the projection from the Veronese surface of degree 4 - but this surface has to be excluded in many situations of classical algebraic geometry) computed by Moishezon, Teicher and others the kernel of (2) actually is trivial. Whether this is a coincidence or a general phenomenon does not seem to be clear.

The author would like to note that he originally believed that the quotient of $\pi_1(X^{\text{aff}}_{\text{gal}})$ he wanted to construct using the algorithm of Zariski and van Kampen was $\mathcal{K}(\pi_1(X^{\text{aff}}), n)$ and so a subgroup of $\pi_1(X^{\text{aff}})^n$. The appearance of (3) and the second homology group was quite some surprise and seems still to be rather mysterious.

One application where it is actually easy to compute the quotient given by (2) is the case when we start with a simply connected surface X. In this case we can also say something about $\pi_1(X_{\text{gal}})$: Namely, suppose that the generic projection is defined by a line bundle $\mathcal{L}$ on X. The degree n of f is precisely the self-intersection number of $\mathcal{L}$. If we denote by d the divisibility index of $\mathcal{L}$ in the Picard group of X then our quotients take the form

$$\begin{array}{rcl} \pi_1(X_{\mathrm{gal}}^{\mathrm{aff}}) & \twoheadrightarrow & {\mathbb{Z}_d}^{n-1} \\ \pi_1(X_{\mathrm{gal}}) & \twoheadrightarrow & {\mathbb{Z}_d}^{n-2}. \end{array}$$

Detailed description of the sections

1 We have a glimpse on fundamental groups of complex algebraic curves and surfaces. After that we give a rather sketchy motivation why complements of divisors on $\mathbb{P}^1$ and $\mathbb{P}^2$ (may) give some insight into the classification problem of algebraic curves and surfaces. Also, generic projections and their Galois closures enter the picture.

2 We introduce the notion of a *good generic projection* that is a little bit more restrictive than the usual notion of a generic projection.

After that we recall some general facts on Galois closures of (good) generic projections. Important for this thesis are the results on the geometry of the ramification loci due to Miyaoka and Faltings.

For a good generic projection $f : X \to \mathbb{P}^2$ we let $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to \mathbb{P}^2$ be its Galois closure. We let ℓ be a generic line in $\mathbb{P}^2$ and let $\mathbb{A}^2$ be the complement $\mathbb{P}^2 - \ell$. Then we define the following objects:

$$\begin{array}{llcll} \text{projective situation:} & f & : & X & \to & \mathbb{P}^2 \\ & f_{\mathrm{gal}} & : & X_{\mathrm{gal}} & \to & \mathbb{P}^2 \\ \text{affine situation:} & f & : & X^{\mathrm{aff}} := X - f^{-1}(\ell) & \to & \mathbb{A}^2 \\ & f_{\mathrm{gal}} & : & X_{\mathrm{gal}}^{\mathrm{aff}} := X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(\ell) & \to & \mathbb{A}^2 \end{array}$$

3 For a given group G and a natural number $n \geq 3$ we define $\mathcal{K}(G,n)$ to be the kernel of the homomorphism from G^n onto G^{ab}. The action of the symmetric group $\mathfrak{S}_n$ on n letters on G^n given by permuting the factors respects $\mathcal{K}(G,n)$. We then form the semidirect product of $\mathcal{K}(G,n)$ by $\mathfrak{S}_n$ via this action:

$$1 \;\to\; \mathcal{K}(G,n) \;\to\; \mathcal{E}(G,n) \;\to\; \mathfrak{S}_n \;\to\; 1.$$

We give some of the basic properties of $\mathcal{K}(G,n)$, prove a universality result, and compute some examples.

4 We describe a certain quotient of the fundamental group of the Galois closure of a good generic projection.

This is done most naturally within the framework of Galois theory. We have therefore given the proof in this setup yielding the result for the étale fundamental group.

Given a generic projection from X of degree n with Galois closure X_{gal} there is a short exact sequence

$$1 \;\to\; \pi_1(X_{\text{gal}}) \;\to\; \pi_1(X_{\text{gal}},\, \mathfrak{S}_n) \;\to\; \mathfrak{S}_n \;\to\; 1 \tag{4}$$

coming from geometry. Here, $\pi_1(X_{\text{gal}}, \mathfrak{S}_n)$ is a generalised fundamental group that classifies covers of X_{gal} together with a $\mathfrak{S}_n$-action.

Using inertia groups we see that this short exact sequence partly splits. Then we take a naturally defined quotient of this exact sequence to force a splitting. Using the universality result for $\mathcal{K}(-,n)$ from Section 3 we then obtain surjective homomorphisms

$$\begin{array}{rcl} \pi_1(X_{\text{gal}}) & \twoheadrightarrow & \mathcal{K}(\pi_1(X),\, n) \\ \pi_1(X_{\text{gal}}^{\text{aff}}) & \twoheadrightarrow & \mathcal{K}(\pi_1(X^{\text{aff}}),\, n). \end{array}$$

This yields a proof of what we have said about the image of (1) above.

To make this proof also work in the topological setup we have to describe how this generalised fundamental group can be defined topologically. To achieve this we use ideas of Grothendieck's [SGA1] and the notion of the orbifold fundamental group.

Having introduced this machinery it is not complicated to carry the results above for the étale fundamental groups over to topological fundamental groups.

5 This is again a purely group theoretical and somewhat technical section which is important for the main results of Section 6.

First we introduce the groups $\mathcal{S}_n(d)$, $d \geq 1$ that generalise the symmetric groups $\mathfrak{S}_n$. These groups should be thought of as symmetric groups with d layers, cf. Section 5.1. It turns out that $\mathcal{S}_n(d)$ for $n \geq 5$ is isomorphic to $\mathcal{E}(\mathfrak{F}_{d-1},\, n)$ where $\mathfrak{F}_{d-1}$ is the free group of rank $d-1$ and where $\mathcal{E}(-,n)$ is as defined in Section 3.

For a finitely generated group G and a natural number $n \geq 3$ we choose a presentation $\mathfrak{F}_d/N$ of G. Using this presentation we construct a quotient of $\mathcal{E}(\mathfrak{F}_d, n)$ that we denote by $\widetilde{\mathcal{E}}(G,n)$. Then we show that this quotient depends only on G and n and not on the presentation chosen. There is a split homomorphism from $\widetilde{\mathcal{E}}(G,n)$ onto $\mathfrak{S}_n$ yielding a split short exact sequence

$$1 \;\to\; \widetilde{\mathcal{K}}(G,n) \;\to\; \widetilde{\mathcal{E}}(G,n) \;\to\; \mathfrak{S}_n \;\to\; 1.$$

This construction is related to the one in Section 3 by a central extension

$$0 \;\to\; H_2(G,\,\mathbb{Z}) \;\to\; \widetilde{\mathcal{K}}(G,n) \;\to\; \mathcal{K}(G,n) \;\to\; 1$$

where $H_2(G,\mathbb{Z})$ denotes the second group homology with coefficients in the integers. Then we give some basic properties of $\widetilde{\mathcal{K}}(G,n)$ and compute it in some cases.

In two appendices we discuss some elementary properties of the second group homology and the connection of $\widetilde{\mathcal{E}}(-,n)$ with some finite and some affine Weyl groups.

6 We first recall the algorithm of Zariski and van Kampen to compute the fundamental group of the complement of a curve in the affine or projective complex plane.

In Section 4 we introduced a certain quotient to split the short exact sequence (4). We show how to use the groups $\mathcal{S}_n(d)$ introduced in Section 5 as a sort of frame when computing this quotient of $\pi_1(X_{\mathrm{gal}}^{\mathrm{aff}})$. Using the isomorphism of $\mathcal{S}_n(d)$ with $\mathcal{E}(\mathfrak{F}_{d-1},n)$ of Section 5 we see that all relations coming from a given generic projection lead exactly to a presentation of $\widetilde{\mathcal{K}}(\pi_1(X^{\mathrm{aff}}),n)$. Hence we obtain a surjective homomorphism

$$\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) \;\twoheadrightarrow\; \widetilde{\mathcal{K}}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}),\,n).$$

The kernel of this map is the one needed to split (4). It is closely related to connectivity results of the inverse image of the ramification locus R_{gal} of f_{gal} in the universal cover of $X_{\mathrm{gal}}^{\mathrm{aff}}$.

Then we study what happens in the projective case. After that we apply our results to generic projections from simply connected surfaces and end this section by some general remarks on symmetric products.

7 In this short section we apply our results to good generic projections from $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, the Hirzebruch surfaces and surfaces in $\mathbb{P}^3$. For generic projections from geometrically ruled surfaces we can compute at least our quotient for the abelianised fundamental group of the Galois closure.

We end this section with the discussion of a sufficiently general projection from the Veronese surface of degree 4 in $\mathbb{P}^5$. Here it is known that the kernel of (2) is non-trivial.

The quotations at the beginning of some of the sections form “Das Trinklied vom Jammer der Erde“ from the Chinese poet Li Bai. The German translation is from Hans Bethge with some minor changes by Gustav Mahler that he used for his “Das Lied von der Erde“.

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1 A short reminder on fundamental groups

Wenn der Kummer naht,
Liegen wüst die Gärten der Seele,
Welkt hin und stirbt die Freude, der Gesang.
Dunkel ist das Leben, ist der Tod.

1.1 Algebraic curves

Let C be a smooth complex projective curve of genus $g(C)$. For $g \geq 1$ we define the following group by generators and relations

$$\Pi_g \;:=\; \langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

Then it is known that

$g(C)$	C is isomorphic to	$\pi_1^{\mathrm{top}}(C)$	
0	$\mathbb{P}^1$	1	
1	an elliptic curve	$\mathbb{Z}^2$	$(= \Pi_1)$
≥ 2	a curve of general type	$\Pi_{g(C)}$	

Also the universal covers of algebraic curves are known: $\mathbb{P}^1$ is homeomorphic to the 2-sphere and its own universal cover. Elliptic curves are the quotient of $\mathbb{R}^2$ by $\mathbb{Z}^2$. Curves of general type are uniformised by the upper half-plane and so their fundamental groups occur as subgroups of $SL_2(\mathbb{R})$.

1.2 Algebraic surfaces

Now let S be a smooth complex projective surface. Since the fundamental group is a birational invariant of smooth varieties we can restrict ourselves to a suitable minimal model of S. We denote by $\kappa(S)$ the Kodaira dimension of S. We recall the Enriques-Kodaira classification (see e.g. [Bea] or [BHPV]):

$\kappa(S)$	S is birational to	$\pi_1^{\mathrm{top}}(S)$
$-\infty$	a $\mathbb{P}^1$-bundle over a smooth curve C	$\pi_1^{\mathrm{top}}(C)$
0	a K3 surface	1
	an Enriques surface	$\mathbb{Z}_2$
	an abelian surface	$\mathbb{Z}^4$
	a bielliptic surface	see below
1	an elliptic surface	see below
2	a surface of general type	unknown in general

We recall that a surface S is called *elliptic* if there exists a flat morphism from S onto a smooth curve C such that the general fibre is a smooth elliptic curve. The singular fibres of such a morphism can be singular curves (nodal or cuspidal rational curves) and they can be multiple.

Now let $S \to C$ be a relatively minimal elliptic surface that has at least one fibre with singular reduction. Suppose there are exactly k multiple fibres above points $P_1, ..., P_k$ of C with multiplicities $m_1, ..., m_k$. By results of Kodaira, Moishezon and Dolgachev there is an isomorphism

$$\pi_1^{\mathrm{top}}(S) \;\cong\; \pi_1^{\mathrm{orb}}(C, \{P_i, m_i\})$$

where π_1^{orb} denotes the orbifold fundamental group (cf. Section 4.4 for a definition of this group). We refer the reader to [Fr, Chapter 7] for details and references.

So if the Kodaira dimension κ of a surface is less than 2 we have some ideas of how its fundamental group looks like. For surfaces of general type the situation is more complicated:

1. Smooth surfaces of degree ≥ 5 in $\mathbb{P}^3$ are simply connected.

2. There are quotients of the latter surfaces by finite groups giving surfaces with finite and non-trivial fundamental groups.

3. If C_1 and C_2 are two curves of genus $g_1 \geq 2$ and $g_2 \geq 2$, respectively then $C_1 \times C_2$ has fundamental group $\Pi_{g_1} \times \Pi_{g_2}$ which is non-abelian and infinite.

At the moment no pattern in the fundamental groups of surfaces of general type is known. Also, it is unclear what these groups can tell us about the classification of surfaces of general type.

We end this section by an example and refer the interested reader to [Hu] for further details and references:

By the Bogomolov-Miyaoka-Yau inequality a minimal surface of general type fulfills $K^2 \leq 9\chi$ where χ denotes the holomorphic Euler characteristic. It is not so complicated to find surfaces with $K^2 \leq 8\chi$ using complete intersections, fibrations or ramified covers. Moreover, Persson [Per] has given examples of minimal surfaces of general type with $\chi = a$ and $K^2 = b$ for almost all admissible pairs (a, b) with $a \leq 8b$.

There where some hints and hopes that surfaces with $K^2 \geq 8\chi$ are uniformised by non-compact domains. Maybe these surfaces were the analogues of the curves of genus ≥ 2 that are uniformised by the upper half-plane? This lead to the so-called "watershed conjecture":

Conjecture 1.1 (Bogomolov et al.) *A surface of general type with $K^2 \geq 8\chi$ has an infinite fundamental group.*

Miyaoka [Mi] gave a construction of surfaces of general type with $K^2 \geq 8\chi$ using Galois closures of generic projections (cf. Section 2.2 for a precise definition). He also showed that every surface has a finite ramified cover that is a surface of general type with $K^2 \geq 8\chi$.

Applying this construction to generic projections from $\mathbb{P}^1 \times \mathbb{P}^1$, Moishezon and Teicher [MoTe1] have shown that there is an infinite number of surfaces of general type with $K^2 \geq 8\chi$ and trivial fundamental group that are not deformation equivalent. In particular, Conjecture 1.1 is false:

Theorem 1.2 (Moishezon-Teicher) *There do exist simply connected surfaces of general type with $K^2 \geq 8\chi$.*

1.3 Complements of branch divisors

Another application of fundamental groups are complements of branch divisors. Some of the following ideas go back to Riemann in the 19th century. We have taken the presentation from [GH, Chapter 2.3]:

Let C be a smooth projective curve of genus $g \geq 2$. Taking the complete linear system to a divisor of degree $n > 2g$ we get an embedding of C into $\mathbb{P}^{n-g}$ as a curve of degree n. Choosing an arbitrary projection onto $\mathbb{P}^1$ (linearly embedded in $\mathbb{P}^{n-g}$) we obtain a ramified cover

$$f \,:\, C \,\to\, \mathbb{P}^1$$

of degree n with a branch divisor $B \subset \mathbb{P}^1$ of degree $2n + 2g - 2$. On the other hand, to give a morphism of degree n from C to $\mathbb{P}^1$ we have to choose a divisor D of degree n on C and a section of $\mathcal{O}_C(D)$. So, at least heuristically, a curve of genus $g \geq 2$ should depend on

$$\begin{array}{rl} & (2n + 2g - 2) \,-\, (n + h^0(C, \mathcal{O}_C(D))) \\ = & (2n + 2g - 2) \,-\, (n + (n - g + 1)) \\ = & 3g - 3 \end{array}$$

parameters - which is in fact the right number.

For $x_0 \in \mathbb{P}^1 - B$ we define a homomorphism

$$\varphi \;:\; \pi_1^{\mathrm{top}}(\mathbb{P}^1 - B,\, x_0) \;\to\; \mathfrak{S}_n$$

where $\mathfrak{S}_n$ is the symmetric group on n letters: We fix a numbering of the n points in the fibre $f^{-1}(x_0)$. If we lift a loop based at x_0 inside $\mathbb{P}^1 - B$ to $C - f^{-1}(B)$ we get a permutation of the points in the fibre and hence an element of $\mathfrak{S}_n$.

We now assume that f is "generic" in the sense that the divisor B consists of $2n+2g-2$ distinct points and that there is no point with ramification index bigger than 2. Knowing B and the homomorphism φ we can reconstruct C out of these data since φ tells us how to cut and glue different copies of $\mathbb{P}^1 - B$ to get C.

The ideas outlined above may generalise in some way to surfaces:

We let $D \subset \mathbb{P}^2$ be the branch divisor of a generic projection $f : S \to \mathbb{P}^2$ of degree $n \geq 5$. In a similar fashion as above we define a homomorphism

$$\pi_1^{\mathrm{top}}(\mathbb{P}^2 - D,\, x_0) \quad \to \quad \mathfrak{S}_n$$

and recover S out of these data by [Ku, Proposition 1]. Moreover, there is even the following

Conjecture 1.3 (Chisini) *Assume that $D \subset \mathbb{P}^2$ is the branch divisor of a generic projection of degree ≥ 5. Then there is a unique generic projection having D as branch divisor.*

For the proof of this conjecture in some important cases and the work of Kulikov and Moishezon on it we refer to [Ku].

There are still discrete invariants missing to distinguish between different components of the moduli space of minimal surfaces of general type with fixed χ and K^2. The results above suggest that it may be possible to get such invariants out of $\pi_1^{\mathrm{top}}(\mathbb{P}^2 - D)$ where D is the branch curve of a generic projection.

So it may be that generic projections turn out to be important for the classification of algebraic surfaces of general type.

2 Generic projections and their Galois closures

Herr dieses Hauses!
Dein Keller birgt die Fülle des goldenen Weins!
Hier, diese Laute nenn' ich mein!
Die Laute schlagen und die Gläser leeren,
das sind die Dinge, die zusammen passen.

2.1 Sufficiently ample line bundles

Let X be a smooth projective surface over the complex numbers.

Definition 2.1 *We call a line bundle $\mathcal{L}$ on X* **sufficiently ample** *if*

1. *$\mathcal{L}$ is an ample line bundle with self-intersection number at least* 5*,*
2. *for every closed point $x \in X$ the global sections $H^0(X, \mathcal{L})$ generate the fibre*
$$\mathcal{L}_x/\mathfrak{m}_x^4 \cdot \mathcal{L},$$
3. *for any pair $\{x, y\}$ of distinct closed points of X the global sections of $\mathcal{L}$ generate the direct sum*
$$\mathcal{L}_x/\mathfrak{m}_x^3 \cdot \mathcal{L} \,\oplus\, \mathcal{L}_y/\mathfrak{m}_y^3 \cdot \mathcal{L},$$
4. *for any triple $\{x, y, z\}$ of distinct closed points of X the global sections of $\mathcal{L}$ generate the direct sum*
$$\mathcal{L}_x/\mathfrak{m}_x^2 \cdot \mathcal{L} \,\oplus\, \mathcal{L}_y/\mathfrak{m}_y^2 \cdot \mathcal{L} \,\oplus\, \mathcal{L}_z/\mathfrak{m}_z^2 \cdot \mathcal{L}.$$

To produce such line bundles later on we will use the following lemma that already appeared as a remark in [Fa, Section 2]:

Lemma 2.2 *If a line bundle is the tensor product of at least five very ample line bundles it is sufficiently ample.*

PROOF. Let $\mathcal{L}_i$, $i = 1, ..., 5$ be very ample line bundles and $\mathcal{M}$ their tensor product. Since the intersection of $\mathcal{L}_i$ with $\mathcal{L}_j$ for all i, j is a positive integer it follows that the self-intersection of $\mathcal{M}$ is at least 25 and so even bigger than 5.

For each closed point $x \in X$ the global sections of each $\mathcal{L}_i$ generate the fibre $\mathcal{L}_{i,x}/\mathfrak{m}_x^2$ since $\mathcal{L}_i$ is very ample. It follows that the global sections of $\mathcal{L}_i \otimes \mathcal{L}_j$ generate the fibre $(\mathcal{L}_i \otimes \mathcal{L}_j)_x/\mathfrak{m}_x^3$ and that the global sections of $\mathcal{L}_i \otimes \mathcal{L}_j \otimes \mathcal{L}_k$ generate the fibre $(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \mathcal{L}_k)_x/\mathfrak{m}_x^4$.

For two distinct closed points $x, y \in X$ there exists a global section of $\mathcal{L}_i$ that is non-zero in $\mathcal{L}_i/\mathfrak{m}_x$ and vanishes in $\mathcal{L}_i/\mathfrak{m}_y$. Using such sections ("partitions of unity") and the results above we see that the global sections of $\mathcal{M}$ fulfill the conditions of Definition 2.1. $\square$

For his conclusions in [Fa] he also needed that $\mathcal{K}_X \otimes \mathcal{L}^{\otimes 3}$ is ample. However, this is automatic under our ampleness assumptions:

Lemma 2.3 *Let $\mathcal{L}$ be a sufficiently ample line bundle. Then the line bundle $\mathcal{K}_X \otimes \mathcal{L}^{\otimes 3}$ is very ample.*

PROOF. This is an application of Reider's theorem [Re]: We denote by L the class of $\mathcal{L}$ in $\mathrm{Pic}X$. By assumption we have $L^2 \geq 4$ and so in particular $(3L)^2 \geq 10$. Since $\mathcal{L}$ is ample every curve E on X fulfills $(3L)E \geq 3$. Now, if the linear system to $\mathcal{K}_X \otimes \mathcal{L}^{\otimes 3}$ had a base point or if it could not separate (possibly infinitely near) points, Reider's theorem would provide us with a curve E such that $(3E)L < 3$ which is impossible. From this we conclude that the line bundle $\mathcal{K}_X \otimes \mathcal{L}^{\otimes 3}$ is very ample. $\square$

Definition 2.4 *We let X be a smooth complex projective surface and we let $\mathcal{L}$ be a sufficiently ample line bundle on X. Let E be a three-dimensional linear subspace of $H^0(X, \mathcal{L})$. We call such a subspace* **generic** *if*

1. *E generates $\mathcal{L}$, i.e. there is a well-defined finite morphism*
$$f = f_E \ : \ X \ \to \ \mathbb{P}(E) = \mathbb{P}^2$$
of degree n equal to the self-intersection number of $\mathcal{L}$,
2. *the ramification locus $R \subset X$ of f_E is a smooth curve,*
3. *the branch locus $D \subset \mathbb{P}^2$ of f_E is a curve with at worst cusps and simple double points as singularities and*
4. *the restriction $f_E|_R : R \to D$ is birational.*

We call the finite morphism f_E associated to a generic three-dimensional linear subspace E a **generic projection**.

We denote by $\mathbb{G}(k, V)$ the Grassmannian parametrising k-dimensional linear subspaces of the vector space V. To justify the name introduced in Definition 2.4 there is the following well-known

Proposition 2.5 *Let $\mathcal{L}$ be a sufficiently ample line bundle on the smooth complex projective surface X. Then there is a dense and open subset G' of $\mathbb{G}(3,\ H^0(X, \mathcal{L}))$ such that all $E \in G'$ are generic and determine generic projections.*

A proof of this proposition under our ampleness assumptions can be found in [Fa, Proposition 1]. $\square$

2.2 Galois closures of generic projections

Let $f : X \to Y$ be a finite morphism between normal surfaces over the complex numbers. Then f induces an extension of the function fields $K(X)/K(Y)$ of degree $n = \deg f$. Let L be the Galois closure of this field extension. Its Galois group is a subgroup of the symmetric group $\mathfrak{S}_n$. Let X_{gal} be the normalisation of X (or, equivalently, of Y) inside L. We denote by $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to Y$ the induced morphism.

Definition 2.6 *Given a finite morphism $f : X \to Y$ between normal surfaces we call the normal surface X_{gal} together with its morphism $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to Y$ the* **Galois closure** *of the morphism $f : X \to Y$.*

Now let X be a smooth complex projective surface and $\mathcal{L}$ be a sufficiently ample line bundle on X with self-intersection number n. The following result is again well-known

Proposition 2.7 *There exists an open dense subset G' of $\mathbb{G}(3, H^0(X, \mathcal{L}))$ such that for all $E \in G'$*

1. *the map f_E associated to E is a generic projection,*
2. *the Galois closure X_{gal} of f_E is a smooth projective surface,*
3. *the Galois group $K(X_{\mathrm{gal}})/K(\mathbb{P}^2)$ is the symmetric group $\mathfrak{S}_n$ and*
4. *the branch curve $D \subset \mathbb{P}^2$ of f_E is an irreducible divisor.*

For a proof of this proposition in our setup we refer to [Fa, Proposition 1]. □

So let X be a smooth projective surface with canonical line bundle $\mathcal{K}_X$ and let $\mathcal{L}$ be a sufficiently ample line bundle on X. We denote by L and K_X the classes of $\mathcal{L}$ and $\mathcal{K}_X$ in $\mathrm{Num}(X)$, respectively.

Proposition 2.8 *Let E be a three-dimensional linear subspace of $H^0(X, \mathcal{L})$ that belongs to the G' of Proposition 2.7. Let $f = f_E$ be the corresponding generic projection of degree $n := L^2$ and let $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to \mathbb{P}^2$ be its Galois closure.*

1. *The branch locus $D \subset \mathbb{P}^2$ of f (and f_{gal}) is an irreducible curve of degree*

$$d \;=\; K_X L + 3L^2 \;=\; K_X L + 3n.$$

2. *This irreducible curve D has δ simple double points and κ cusps, where*

$$\begin{array}{rcl} \delta &=& d^2/2 - 15d + 24n - 4K_X^2 + 12\chi(\mathcal{O}_X) \\ \kappa &=& 9d - 15n + 3K_X^2 - 12\chi(\mathcal{O}_X) \quad . \end{array}$$

For a proof we refer to [Fa, Section 4] or [MoTe1, Chapter 0]. □

Since the existence of singularities on the branch curve of a generic projection plays an important rôle later on we remark that

Lemma 2.9 *Let $f : X \to \mathbb{P}^2$ be a generic projection given by a sufficiently ample line bundle $\mathcal{L}$.*

1. *There is at least one cusp on the branch curve of f.*
2. *There exists a positive integer m_0 (depending on $\mathcal{L}$) such that for all $m \geq m_0$ the branch curve of a generic projection with respect to $\mathcal{L}^{\otimes m}$ has at least one simple double point.*

PROOF. The degree of a generic projection f is equal to the self-intersection number of $\mathcal{L}$ which is at least 5. By a theorem of Gaffney and Lazarsfeld (quoted as [FL, Theorem 6.1]) there exists a closed point x on X with ramification index at least 3. The image $f(x)$ of x lies on the branch curve D of f and D necessarily has a cusp in such a point.

The number of simple double points of a branch curve of a generic projection with respect to the line bundle $\mathcal{L}^{\otimes m}$ is a polynomial of degree 4 as a function of m tending to $+\infty$ as m tends to $+\infty$. Hence there exists a positive integer m_0 as stated above. □

2.3 Questions on connectivity

Definition 2.10 *We let $\mathfrak{S}_n$ be the symmetric group on n letters. Then we denote its subgroup of permutations fixing the letter i by $\mathbf{\mathfrak{S}_{n-1}^{(i)}}$.*

Definition 2.11 *For a permutation of $\mathfrak{S}_n$ we define its* **support** *to be the largest subset of $\{1, ..., n\}$ on which it acts non-trivially. We say that two permutations are* **disjoint** *or* **nodal** *if their supports are disjoint. In the case where their supports intersect in exactly one element we say that they are* **cuspidal**.

We let $\mathcal{L}$ be a sufficiently ample line bundle on the smooth projective surface X. We let E be a three-dimensional linear subspace of $H^0(X, \mathcal{L})$ belonging to the G' given by Proposition 2.7. We let $f = f_E : X \to \mathbb{P}^2$ be the corresponding generic projection of degree n and denote by $f_{\text{gal}} : X_{\text{gal}} \to \mathbb{P}^2$ its Galois closure.

We denote by $R_{\text{gal}} \subset X_{\text{gal}}$ the ramification divisor of f_{gal}. We know from Proposition 2.7 that the symmetric group $\mathfrak{S}_n$ acts on X_{gal}. For a transposition τ of $\mathfrak{S}_n$ we consider the following components of R_{gal}:

$$R_\tau \ := \ \text{Fix}(\tau) \ := \ \{x \in X_{\text{gal}},\, \tau x = x\}.$$

Then there is the following result

Proposition 2.12 *Let $\mathcal{L}$ be a sufficiently ample line bundle on X and let $f = f_E : X \to \mathbb{P}^2$ be a generic projection coming from a three-dimensional linear subspace $E \in G'$ with G' as in Proposition 2.7. We furthermore assume that the branch curve of f has a simple double point. Then*

1. *The R_τ's defined above are smooth and irreducible curves.*
2. *The ramification locus R_{gal} of f_{gal} is the union of the R_τ's where τ runs through the transpositions of $\mathfrak{S}_n$.*
3. *If τ_1 and τ_2 are disjoint transpositions then R_{τ_1} and R_{τ_2} intersect transversely. These intersection points lie over simple double points of D and there is no other component of R_{gal} through such points.*
4. *If τ_1 and τ_2 are cuspidal transpositions then R_{τ_1} and R_{τ_2} intersect transversely. These intersection points lie over cusps of D and the only other component of R_{gal} through such points is $R_{\tau_1\tau_2\tau_1^{-1}} = R_{\tau_2\tau_1\tau_2^{-1}}$.*

For a proof we refer to [Fa, Lemma 1] and [Fa, Section 4]. We note that a less precise statement without proof was already made by Miyaoka [Mi]. □

Definition 2.13 *Let $\mathcal{L}$ be a sufficiently ample line bundle on a smooth projective surface X. We call a generic projection $f = f_E : X \to \mathbb{P}^2$ associated to a three-dimensional linear subspace $E \in G'$ with G' as in Proposition 2.7 a* **good generic projection** *if the branch curve of f has a simple double point.*

By Lemma 2.2 the tensor product of five very ample line bundles is sufficiently ample. Twisting a sufficiently ample line bundle with itself at least m_0 times with m_0 as in Lemma 2.9 we arrive at a line bundle $\mathcal{L}'$ such that there is an open dense subset of $\mathbb{G}(3, H^0(X, \mathcal{L}'))$ giving rise to good generic projections.

It is in this sense that a "sufficiently general" three-dimensional linear subspace of the space of global sections of an ample line bundle gives rise to a good generic projection for "almost all" ample line bundles.

We let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n with Galois closure $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to \mathbb{P}^2$. Let ℓ be a generic line in $\mathbb{P}^2$, i.e. a line intersecting D in $\deg D$ distinct points. We then define

$$\begin{aligned} \mathbb{A}^2 &:= \mathbb{P}^2 - \ell, \\ X^{\mathrm{aff}} &:= f^{-1}(\mathbb{A}^2), \\ X^{\mathrm{aff}}_{\mathrm{gal}} &:= {f_{\mathrm{gal}}}^{-1}(\mathbb{A}^2). \end{aligned}$$

We let $p : Y^{\mathrm{aff}} \to X^{\mathrm{aff}}_{\mathrm{gal}}$ be a topological cover of $X^{\mathrm{aff}}_{\mathrm{gal}}$ or $p : Y \to X_{\mathrm{gal}}$ be a topological cover of X_{gal}. Then for all transpositions τ of $\mathfrak{S}_n$ the inverse image $p^{-1}(R_\tau)$ is a disjoint union of smooth and irreducible curves.

Question 2.14 *Is it true that for distinct transpositions τ_1 and τ_2 every irreducible component of $p^{-1}(R_{\tau_1})$ intersects every irreducible component of $p^{-1}(R_{\tau_2})$?*

In this thesis we want to compute the fundamental groups $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}})$ and $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})$. The main result (Theorem 6.2) is that there is always a surjective homomorphism

$$\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) \quad \twoheadrightarrow \quad \widetilde{\mathcal{K}}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}),\, n)$$

where $\widetilde{\mathcal{K}}(-, n)$ is the group-theoretic construction defined in Section 5.3.

Now, if Question 2.14 has an affirmative answer for all topological covers of $X_{\mathrm{gal}}^{\mathrm{aff}}$ then the kernel of this surjective homomorphism is trivial.

3 Semidirect products by symmetric groups

Ein voller Becher Weins zur rechten Zeit
Ist mehr wert, als alle Reiche dieser Erde!
Dunkel ist das Leben, ist der Tod.

3.1 Definition of $\mathcal{K}(-,\mathrm{n})$ and $\mathcal{E}(-,\mathrm{n})$

Let G be an arbitrary group and $n \geq 2$ be a natural number. We denote by θ the permutation representation of the symmetric group $\mathfrak{S}_n$ on G^n given by

$$\begin{array}{rcccc} \theta & : & \mathfrak{S}_n & \to & \mathrm{Aut}(G^n) \\ & & \sigma & \mapsto & \big(\theta(\sigma) : (g_1, ..., g_n) \mapsto (g_{\sigma^{-1}(1)}, ..., g_{\sigma^{-1}(n)})\big) \end{array}$$

Then we form the split extension of groups with respect to θ

$$1 \;\to\; G^n \;\to\; G^n \rtimes_\theta \mathfrak{S}_n \;\to\; \mathfrak{S}_n \;\to\; 1$$

and denote by $s : \mathfrak{S}_n \to G^n \rtimes_\theta \mathfrak{S}_n$ the associated splitting.

We define the subgroup $\mathcal{E}(G, n)$ of $G^n \rtimes_\theta \mathfrak{S}_n$ to be the group generated by all conjugates of $s(\mathfrak{S}_n)$ and define $\mathcal{K}(G, n)$ to be the intersection $G^n \cap \mathcal{E}(G, n)$. Hence we get a split extension

$$1 \;\to\; \mathcal{K}(G,\, n) \;\to\; \mathcal{E}(G,\, n) \;\to\; \mathfrak{S}_n \;\to\; 1.$$

We give another characterisation of these groups in Proposition 3.3.

More generally, let S be a subgroup of $\mathfrak{S}_n$. Then we define

$$\begin{array}{rclcl} \mathcal{E}(G,\, n)_S & := & \langle \vec{g} s(\sigma) \vec{g}^{-1} \,|\, \vec{g} \in G^n, \sigma \in S \rangle & \leq & \mathcal{E}(G,\, n) \\ \mathcal{K}(G,\, n)_S & := & \mathcal{E}(G,\, n)_S \cap \mathcal{K}(G,\, n) & \trianglelefteq & \mathcal{K}(G,\, n) \quad . \end{array}$$

These subgroups remain the same when passing to a G^n-conjugate splitting. We will therefore suppress s in future. Clearly, $\mathcal{K}(G, n)_S$ is always a normal subgroup of G^n and $\mathcal{K}(G, n)$.

In the notation introduced in Definition 2.10 we have the following equalities and isomorphisms:

$$\begin{array}{rcll} \mathcal{E}(G,\, n)_{\mathfrak{S}_n} & = & \mathcal{E}(G,\, n) & \\ \mathcal{E}(G,\, n)_{\mathfrak{S}^{(i)}_{n-1}} & \cong & \mathcal{E}(G,\, n-1) & \text{for } n \geq 3 \end{array}$$

and similarly for $\mathcal{K}(-, n)$.

Later on we have to deal with subgroups of $\mathcal{K}(G, n)$ that are generated by $\mathcal{K}(G, n)$-conjugates of a subgroup S of $\mathfrak{S}_n$ rather than G^n-conjugates. Fortunately, we have the following

Lemma 3.1 *Let S be a subgroup of $\mathfrak{S}_n$, $n \geq 3$ that is generated by transpositions. Then*

$$\begin{aligned}
\mathcal{E}(G,\, n)_S &\stackrel{\text{def}}{=} \langle \vec{g}\,\sigma\,\vec{g}^{-1} \,|\, \vec{g} \in G^n,\, \sigma \in S\rangle \\
&= \langle \vec{g}\,\sigma\,\vec{g}^{-1} \,|\, \vec{g} \in \mathcal{K}(G,\, n),\, \sigma \in S\rangle \\
\mathcal{K}(G,\, n)_S &\stackrel{\text{def}}{=} \langle [\vec{g},\, \sigma] \,|\, \vec{g} \in G^n, \sigma \in S\rangle \\
&= \langle [\vec{g},\, \sigma] \,|\, \vec{g} \in \mathcal{K}(G,\, n), \sigma \in S\rangle.
\end{aligned}$$

Moreover, it is enough that σ runs through a system of generating transpositions of S in the expressions above.

PROOF. We will first assume that $S = \langle \tau \rangle$ for the transposition $\tau = (1\,2)$ of $\mathfrak{S}_n$. For $(g_1, ..., g_n) \in G^n$ we calculate

$$(g_1, ..., g_n)\tau(g_1, ..., g_n)^{-1} = (g_1 {g_2}^{-1}, g_2 {g_1}^{-1}, 1, ..., 1)\tau.$$

In this case the subgroup $\mathcal{K}(G, n)_S$ of G^n is generated by $(g, g^{-1}, 1, ..., 1)$, $g \in G$. Since we assumed $n \geq 3$ we may consider the element $(g, 1, g^{-1}, 1, ...)$. By applying the previous calculation to the transposition $(1\,3)$ this is also an element of $\mathcal{K}(G, n)$. From

$$(g, 1, g^{-1}, 1, ..., 1)\tau(g, 1, g^{-1}, 1, ..., 1)^{-1} = (g, g^{-1}, 1, ..., 1)\tau$$

we deduce that $\langle \vec{g}\sigma\vec{g}^{-1} \,|\, \vec{g} \in \mathcal{K}(G, n), \sigma \in S\rangle$ is generated by the same elements as $\mathcal{E}(G, n)_S$. So both subgroups are equal. A similar calculation yields the result for $\mathcal{K}(G, n)_S$.

We now let S be a subgroup of $\mathfrak{S}_n$ generated by transpositions. Then we can write $\sigma \in S$ as a product $\tau_1 \cdot ... \cdot \tau_d$ of transpositions all lying in S. For $\vec{g} \in G^n$ we get

$$\vec{g}\sigma\vec{g}^{-1} = \vec{g}\left(\prod_{i=1}^{d} \tau_i\right)\vec{g}^{-1} = \prod_{j=i}^{d} \vec{g}\tau_i\vec{g}^{-1}.$$

We have seen above that all $\vec{g}\tau_i\vec{g}^{-1}$ can be written as products of of conjugates of τ_i under $\mathcal{K}(G, n)$. So $\vec{g}\sigma\vec{g}^{-1}$ can be written as a product of $\mathcal{K}(G, n)$-conjugates of elements of S.

To prove the remaining assertion we assume that $\sigma \in S$ can be written as a product of d transpositions of S. The case $d = 1$ was already done above. We can find a transposition τ and an element ν that can be written as a product of strictly less that d transpositions such that $\sigma = \tau \cdot \nu$. Then also $\tau\nu\tau^{-1}$ can be written as a product of strictly less than d transpositions and writing

$$[\vec{g},\, \tau\nu] = [\vec{g}, \tau] \,\cdot\, \tau[\vec{g}, \nu]\tau^{-1} = [\vec{g}, \tau] \,\cdot\, [\tau\vec{g}\tau^{-1}, \tau\nu\tau^{-1}]$$

we can apply induction. □

Remark 3.2 *The assumption $n \geq 3$ is crucial:*

1. *If $n = 1$ then $\mathcal{E}(G, 1) = \mathcal{K}(G, 1) = 1$.*
2. *If $n = 2$ then $\mathcal{K}(G,\, 2)$ is the subgroup of G^2 generated by (g, g^{-1}) and*

$$\begin{array}{lcl} \langle \vec{g}\sigma\vec{g}^{-1} \,|\, \vec{g} \in G^2, \sigma \in \mathfrak{S}_2 \rangle & = & \langle (g, g^{-1}), g \in G \rangle \;=\; \mathcal{K}(G,\, 2) \\ \langle \vec{g}\sigma\vec{g}^{-1} \,|\, \vec{g} \in \mathcal{K}(G,\, 2), \sigma \in \mathfrak{S}_2 \rangle & = & \langle (g^2, g^{-2}), g \in G \rangle \;\leq\; \mathcal{K}(G,\, 2). \end{array}$$

   So in this case it depends on the structure of G whether these two subgroups coincide.

3.2 Properties

There exists a quite different description of $\mathcal{K}(-, n)$ given by the following

Proposition 3.3 *Let $n \geq 2$ be a natural number and G be an arbitrary group. Then*

$$\mathcal{K}(G, n) \;=\; \ker \; \left(\begin{array}{ccc} G^n & \to & G^{\mathrm{ab}} \\ (g_1, ..., g_n) & \mapsto & g_1 \cdot ... \cdot g_n \end{array} \right)$$

as subgroups of G^n.

PROOF. Lemma 3.1 tells us that $\mathcal{K}(G, n)$ is generated by elements of the form $(1, ..., 1, g, 1, ..., 1, g^{-1}, 1, ..., 1)$. Since these elements lie in the kernel of the map $G^n \to G^{\mathrm{ab}}$ it follows that we already have $\mathcal{K}(G, n) \leq \ker(G^n \to G^{\mathrm{ab}})$.

Conversely, suppose that $(g_1, ..., g_n)$ lies in the kernel of $G^n \to G^{\mathrm{ab}}$. Multiplying by $(1, ..., g_n, {g_n}^{-1})$ we obtain an element of the form $(g_1, ..., g'_{n-1}, 1)$. We multiply this element by $(1, ..., 1, g'_{n-1}, {g'_{n-1}}^{-1}, 1)$. Proceeding inductively, we see that every element of $\ker(G^n \to G^{\mathrm{ab}})$ can be changed by elements from $\mathcal{K}(G, n)$ to an element of the form $(g'_1, 1, ..., 1)$. Then necessarily $g'_1 \in [G, G]$. This means that g'_1 is a product of commutators. Since $n \geq 2$ we can write a commutator as a product of elements of $\mathcal{K}(G, n)$:

$$\begin{array}{ll} & [(h_1, 1, ..., 1),\, (h_2, 1, ..., 1)] \\ = & (h_1, {h_1}^{-1}, 1, ..., 1)\, (h_2, {h_2}^{-1}, 1, ..., 1)\, ((h_2 h_1)^{-1}, (h_2 h_1), 1, ..., 1). \end{array}$$

For computations later on we remark that for $n \geq 3$ such a commutator is even a commutator of elements of $\mathcal{K}(G, n)$:

$$[(h_1, 1, ..., 1),\, (h_2, 1, ..., 1)] = [(h_1, {h_1}^{-1}, 1, ...),\, (h_2, 1, {h_2}^{-1}, 1, ...)].$$

Hence every element of $\ker(G^n \to G^{\mathrm{ab}})$ is a product of elements of $\mathcal{K}(G, n)$. This proves the converse inclusion and so we are done. □

The following proposition provides us with a couple of short exact sequences. They turn out to be useful for calculations later on.

Proposition 3.4 *Let $n \geq 2$ be a natural number. We denote by p_i the projection from G^n onto its i.th factor.*

1. *Let τ be a transposition that moves the index i. Then p_i induces a surjective homomorphism*
$$\mathcal{K}(G,\, n)_{\langle\tau\rangle} \stackrel{p_i}{\twoheadrightarrow} G.$$
If G is abelian then this is an isomorphism of groups.

2. *For $n \geq 3$ the projection p_i induces a short exact sequence*
$$1 \;\rightarrow\; \mathcal{K}(G,\, n)_{\mathfrak{S}^{(i)}_{n-1}} \;\rightarrow\; \mathcal{K}(G,\, n) \;\stackrel{p_i}{\rightarrow}\; G \;\rightarrow\; 1.$$
This allows us to recover G from $\mathcal{K}(G, n)$.

3. *Combining the projections $\pi := p_2 \times \ldots \times p_n$ we obtain an exact sequence*
$$1 \;\rightarrow\; [G, G] \;\rightarrow\; \mathcal{K}(G,\, n) \;\stackrel{\pi}{\rightarrow}\; G^{n-1} \;\rightarrow\; 1.$$
In general this exact sequence is not split as Example 3.11 shows.

PROOF. We will assume $i = 2$ and $\tau = (1\,2) \in \mathfrak{S}_n$. We have already seen in the proof of Lemma 3.1 that $\mathcal{K}(G, n)_{\langle\tau\rangle}$ equals to the subgroup of G^n generated by $(g^{-1}, g, 1, ..., 1)$. (For this statement it suffices to assume that $n \geq 2$.) The projection p_2 of G^n onto its second factor induces a surjection of $\mathcal{K}(G, n)_{\langle\tau\rangle}$ onto G. We can split this projection by the (set-theoretical) map

$$\begin{array}{ccc} G & \rightarrow & \mathcal{K}(G, n)_{\langle\tau\rangle} \\ g & \mapsto & (g^{-1}, g, 1, ..., 1) \end{array}$$

proving the surjectivity of p_2. If G is abelian then $g \mapsto g^{-1}$ is a homomorphism. In this case p_2 and its splitting are isomorphisms.

From Proposition 3.3 we conclude that $\ker p_i \cap \mathcal{K}(G, n)$ is equal to the kernel of the restriction of $G^n \rightarrow G^{\mathrm{ab}}$ to $\ker p_i = [G^n, \mathfrak{S}^{(i)}_{n-1}]$. But this is precisely $[\mathcal{K}(G, n), \mathfrak{S}^{(i)}_{n-1}]$ proving the second exact sequence.

Given an element $(g_2, ..., g_n) \in \pi(G^n)$ we set $g_1 := (g_2 \cdot \ldots \cdot g_n)^{-1}$. By Proposition 3.3 the element $(g_1, g_2, ..., g_n)$ lies in $\mathcal{K}(G, n)$ and maps to $(g_2, ..., g_n)$ under π. This proves that π is surjective. An element of $\ker \pi$ is of the form $(g_1, 1, ..., 1)$ and if this element also lies in $\mathcal{K}(G, n)$ Proposition 3.3 tells us that $g_1 \in [G, G]$. On the other hand, given an element $g_1 \in [G, G]$ then $(g_1, 1, ..., 1)$ lies in $\mathcal{K}(G, n)$ by Proposition 3.3 and also in the kernel of π. This is enough to prove the third short exact sequence. □

Corollary 3.5 *Let $n \geq 2$.*

1. *If G is abelian, i.e. $[G,G]=1$, then $\mathcal{K}(G,\, n) \cong G^{n-1}$.*
2. *If G is perfect, i.e. $[G,G]=G$, then $\mathcal{K}(G,\, n) = G^n$.*

PROOF. Let G be abelian. Then we define a map from G^{n-1} to G^n via

$$\begin{array}{ccc} G^{n-1} & \to & G^n \\ (g_1, ..., g_{n-1}) & \mapsto & (g_1, ..., g_{n-1}, (g_1 \cdot ... \cdot g_{n-1})^{-1}). \end{array}$$

Since G is abelian this defines a homomorphism of groups. Clearly, it is injective. The image of G^{n-1} lies inside $\mathcal{K}(G,n)$ by Proposition 3.3. Also all elements of the form $(1,...,g,1,..,1,g^{-1},1,...)$ lie in the image. By Lemma 3.1 these elements generate $\mathcal{K}(G,n)$ and so this homomorphism is surjective.

If G is perfect then $G^{\mathrm{ab}}=1$ and so $\ker(G^n \to G^{\mathrm{ab}}) = G^n$. □

We denote by $\mathcal{P}_n$ the permutation representation of $\mathfrak{S}_n$ on $\mathbb{Z}^n$. This is the same as the representation induced from the trivial representation of $\mathfrak{S}_{n-1}^{(1)}$. Inside $\mathcal{P}_n$ we form the direct sum of $\mathbb{Z}(1,...,1)$ with trivial $\mathfrak{S}_n$-action and the $\mathfrak{S}_n$-stable hyperplane

$$\tilde{\mathcal{P}}_n \;=\; \left\{(k_1,...,k_n) \in \mathbb{Z}^n \mid \textstyle\sum_{i=1}^n k_i = 0\right\}.$$

After tensoring with $\mathbb{Q}$ this defines a decomposition of $\mathcal{P}_n \otimes_{\mathbb{Z}} \mathbb{Q}$ as direct sum of the trivial representation and the irreducible representation $\tilde{\mathcal{P}}_n \otimes_{\mathbb{Z}} \mathbb{Q}$.

From Proposition 3.3 we then get the following description of $\mathcal{K}(-,n)$ for abelian groups in terms of the representation theory of the symmetric group $\mathfrak{S}_n$:

Corollary 3.6 *There exists an isomorphism of $\mathbb{Z}[\mathfrak{S}_n]$-modules*

$$\mathcal{K}(\mathbb{Z},\, n) \;\cong\; \tilde{\mathcal{P}}_n.$$

Moreover, for every abelian group G there is a $\mathfrak{S}_n$-equivariant isomorphism

$$\mathcal{K}(G,\, n) \;\cong\; G \otimes_{\mathbb{Z}} \tilde{\mathcal{P}}_n.$$

The following corollary shows that $\mathcal{K}(-,n)$ inherits many of the properties of the group we plug in:

Corollary 3.7 *Let $n \geq 2$ and consider the following properties of groups:*

abelian, finite, nilpotent, perfect, solvable.

Then G has one of the properties above if and only if $\mathcal{K}(G,n)$ has the respective property.

PROOF. If G is abelian (resp. finite, nilpotent, solvable) then so is $\mathcal{K}(G,n)$ being a subgroup of G^n. If G is perfect then $\mathcal{K}(G,n) \cong G^n$ which is perfect.

By the first exact sequence of Proposition 3.4 there exists a surjective homomorphism from $\mathcal{K}(G,n)$ onto G. So if $\mathcal{K}(G,n)$ is abelian (resp. finite, nilpotent, perfect, solvable) then so is G being a quotient of $\mathcal{K}(G,n)$. $\square$

We finally give some basic functoriality properties of our construction:

Proposition 3.8 *Let $n \geq 2$ be a natural number and let G_1, G_2, G be arbitrary groups.*

1. *If $G_1 \to G_2$ is an injection then so is $\mathcal{K}(G_1, n) \to \mathcal{K}(G_2, n)$.*
2. *If $G_1 \to G_2$ is a surjection then so is $\mathcal{K}(G_1, n) \to \mathcal{K}(G_2, n)$.*
3. *If G is a semidirect product then so is $\mathcal{K}(G,n)$. However, the functor $\mathcal{K}(-,n)$ is not exact in the middle as Example 3.11 and Example 3.13 show.*
4. *$\mathcal{K}(G_1 \times G_2, n) = \mathcal{K}(G_1, n) \times \mathcal{K}(G_2, n)$.*
5. *If G is an abelian group then*
$$\begin{array}{rcl} \mathcal{K}(G,\, n)_{\mathrm{tors}} & \cong & \mathcal{K}(G_{\mathrm{tors}},\, n) \\ \mathcal{K}(G,\, n) \otimes_{\mathbb{Z}} \mathbb{Q} & \cong & \mathcal{K}(G \otimes_{\mathbb{Z}} \mathbb{Q},\, n) \end{array}$$
where $-_{\mathrm{tors}}$ denotes the torsion subgroup of an abelian group.
6. *For $n \geq 3$ the natural homomorphism from $\mathcal{K}(G,n)^{\mathrm{ab}}$ onto $\mathcal{K}(G^{\mathrm{ab}}, n)$ is an isomorphism. The assumption $n \geq 3$ is needed as Example 3.12 shows.*

PROOF. We assume that $1 \to K \to G \to Q \to 1$ is exact. Then also the induced sequence $1 \to K^n \to G^n \to Q^n \to 1$ is exact. This induces homomorphisms (notation as in the beginning of this section)

$$K^n \rtimes_\theta \mathfrak{S}_n \quad \to \quad G^n \rtimes_\theta \mathfrak{S}_n \quad \to \quad Q^n \rtimes_\theta \mathfrak{S}_n$$

and induces injections $\mathcal{E}(K,n) \hookrightarrow \mathcal{E}(G,n)$ and $\mathcal{K}(K,n) \hookrightarrow \mathcal{K}(G,n)$. This proves the first assertion (we do not need the normality of K in G in this step). The group $\mathcal{E}(Q,n)$ is generated by $\mathfrak{S}_n$ and commutators $[q,\sigma]$, $q \in Q$. Since $G \twoheadrightarrow Q$ is surjective we see that $\mathcal{E}(G,n) \twoheadrightarrow \mathcal{E}(Q,n)$ is surjective since we can lift elements of $\mathfrak{S}_n$ and commutators. Similarly we see that $\mathcal{K}(G,n) \twoheadrightarrow \mathcal{K}(Q,n)$ is surjective.

If G is a semidirect product then there exists a split surjection $G \twoheadrightarrow Q$. This map induces a split surjection $\mathcal{K}(G,n) \twoheadrightarrow \mathcal{K}(Q,n)$. Therefore also $\mathcal{K}(G,n)$ is a semidirect product.

The assertions about the torsion and the free part of an abelian group follow immediately from Corollary 3.5.

The surjection $G \twoheadrightarrow G^{\mathrm{ab}}$ and the universal property of abelianisation imply that there is a natural surjective homomorphism $\mathcal{K}(G,n)^{\mathrm{ab}} \twoheadrightarrow \mathcal{K}(G^{\mathrm{ab}},n)$:

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{K}(G,n) & \to & G^n & \to & G^{\mathrm{ab}} & \to & 1 \\ & & \downarrow & & \downarrow^{\mathrm{ab}} & & \| & & \\ 1 & \to & \mathcal{K}(G^{\mathrm{ab}},n) & \to & (G^{\mathrm{ab}})^n & \to & G^{\mathrm{ab}} & \to & 1 \end{array} .$$

An element of the kernel $\mathcal{K}(G,n) \to \mathcal{K}(G^{\mathrm{ab}},n)$ is also an element of the kernel of $G^n \to (G^{\mathrm{ab}})^n$ which is $[G,G]^n$. Since we assumed $n \geq 3$ every commutator $(1,...,1,[h_1,h_2],1...,1)$ lies not only in $\mathcal{K}(G,n)$ but is even a commutator of elements of $\mathcal{K}(G,n)$, cf. the proof of Proposition 3.3. This implies that the kernel of $\mathcal{K}(G,n) \to \mathcal{K}(G,n)^{\mathrm{ab}}$ is the commutator subgroup of $\mathcal{K}(G,n)$. Hence the canonical homomorphism from $\mathcal{K}(G,n)^{\mathrm{ab}}$ onto $\mathcal{K}(G^{\mathrm{ab}},n)$ is an isomorphism for $n \geq 3$. □

3.3 Universality

We assume that we are given a group X and a homomorphism $\varphi : \mathfrak{S}_n \to \mathrm{Aut}(X)$ with $n \geq 3$. Then we form the semidirect product

$$1 \;\to\; X \;\to\; X \rtimes_\varphi \mathfrak{S}_n \;\to\; \mathfrak{S}_n \;\to\; 1.$$

We consider $\mathfrak{S}_n$ as a subgroup of the group in the middle via the associated splitting. For a subgroup $S \leq \mathfrak{S}_n$ we denote $[X,S]$ by X_S. Again, X_S is a normal subgroup of X and does not change if we pass to an X-conjugate splitting.

Proposition 3.9 *Let $\varphi : \mathfrak{S}_n \to \mathrm{Aut}(X)$, $n \geq 3$ be a homomorphism and let*

$$1 \;\to\; X \;\to\; X \rtimes_\varphi \mathfrak{S}_n \;\to\; \mathfrak{S}_n \;\to\; 1$$

be the split extension determined by φ. If we define

$$Y \;:=\; X_{\mathfrak{S}_n} / X_{\mathfrak{S}_{n-1}^{(1)}}$$

then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & X_{\mathfrak{S}_n} & \to & X_{\mathfrak{S}_n} \rtimes_\varphi \mathfrak{S}_n & \to & \mathfrak{S}_n & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathcal{K}(Y,n) & \to & \mathcal{K}(Y,n) \rtimes_\theta \mathfrak{S}_n & \to & \mathfrak{S}_n & \to & 1 \end{array}$$

where all homomorphisms downwards are surjective. Moreover, we have an exact sequence

$$1 \;\to\; \bigcap_{i=1}^n X_{\mathfrak{S}_{n-1}^{(i)}} \;\to\; X_{\mathfrak{S}_n} \;\to\; \mathcal{K}(Y,n) \;\to\; 1.$$

PROOF. We will consider the projections $p_i : X_{\mathfrak{S}_n} \to X_{\mathfrak{S}_n}/X_{\mathfrak{S}_{n-1}^{(i)}}$.

Every $\sigma \in \mathfrak{S}_n$ induces via conjugation an isomorphism

$$\begin{array}{ccc} X_{\mathfrak{S}_n}/X_{\mathfrak{S}_{n-1}^{(i)}} & \to & X_{\mathfrak{S}_n}/X_{\mathfrak{S}_{n-1}^{(\sigma(i))}} \\ x & \mapsto & \sigma x \sigma^{-1} \end{array}$$

We claim that this isomorphism depends only on the coset $\mathfrak{S}_n/\mathfrak{S}_{n-1}^{(i)}$:

If $\sigma \in \mathfrak{S}_{n-1}^{(i)}$ then we compute

$$\begin{array}{rcl} x \mod X_{\mathfrak{S}_{n-1}^{(i)}} & \mapsto & \sigma x \sigma^{-1} \mod X_{\mathfrak{S}_{n-1}^{(i)}} \\ & = & x[x^{-1}, \sigma] \mod X_{\mathfrak{S}_{n-1}^{(i)}} \\ & = & x \mod X_{\mathfrak{S}_{n-1}^{(i)}} \end{array}$$

So in this case, the induced automorphism is just the identity and we will identify the different quotients $X_{\mathfrak{S}_n}/X_{\mathfrak{S}_{n-1}^{(i)}}$ via these isomorphisms in the sequel.

Combining the different projections p_i, we obtain a homomorphism

$$p \ : \ X_{\mathfrak{S}_n} \ \to \ \prod_{i=1}^n X_{\mathfrak{S}_n}/X_{\mathfrak{S}_{n-1}^{(i)}} \ \cong \ Y^n$$

with kernel $\ker p = \bigcap_{i=1}^n X_{\mathfrak{S}_{n-1}^{(i)}}$.

We will show that the $\mathfrak{S}_n$-action on $X_{\mathfrak{S}_n}$ via φ is compatible via the projection p with the $\mathfrak{S}_n$-action on Y via θ as described in Section 3.1: Let $\sigma \in \mathfrak{S}_n$ and $x \in X$. Then we calculate

$$\begin{array}{rcl} p_i(\varphi(\sigma)(x)) & = & \varphi(\sigma)(x) \mod X_{\mathfrak{S}_{n-1}^{(i)}} \\ & = & \sigma x \sigma^{-1} \mod X_{\mathfrak{S}_{n-1}^{(i)}} \\ & = & x \mod \sigma^{-1} X_{\mathfrak{S}_{n-1}^{(i)}} \sigma \\ & = & p_{\sigma^{-1}(i)}(x) \\ & = & \theta(\sigma) p_i(x), \end{array}$$

i.e. the homomorphism p is $\mathfrak{S}_n$-equivariant.

By the same calculations as in the proof of Lemma 3.1 we see that $X_{\mathfrak{S}_n}$ is generated by elements $[x, \tau] = x \cdot \varphi(\tau)(x^{-1})$ where x runs through X and τ runs through the transpositions of $\mathfrak{S}_n$. Hence the image of p is generated by the elements $[p(x), \tau] = p(x) \cdot \theta(\tau)(p(x)^{-1})$ and therefore lies inside $\mathcal{K}(Y, n)$.

We define $\tau_i := (1i)$ for $i \geq 2$. Then the second isomorphism theorem of groups yields

$$\frac{X_{\tau_i}}{X_{\tau_i} \cap X_{\mathfrak{S}_{n-1}^{(i)}}} \ = \ \frac{X_{\mathfrak{S}_n}}{X_{\mathfrak{S}_{n-1}^{(i)}}} \ \cong \ Y.$$

For $j \neq i$, $j \geq 2$ the group X_{τ_j} is a subgroup of $X_{\mathfrak{S}_{n-1}^{(i)}}$ so we see that we can in fact construct a surjective homomorphism

$$p_2 \times ... \times p_n \quad : \quad X_{\mathfrak{S}_n} \quad \twoheadrightarrow \quad Y^{n-1}.$$

We have already seen in the proof of Lemma 3.1 that $\mathcal{K}(Y, n)$ is generated by the elements $[(1, ..., 1, y, 1, ...), \tau]$ with $y \in Y$.

So let $\tau = (2\,3)$ and $y \in Y$ be arbitrary. By what we have just proved and using $n \geq 3$ we can find $x \in X$ with $p_2(x) = y$ and $p_3(x) = 1$. Then

$$p_i([x, \tau]) = p_i(x \cdot \varphi(\tau)(x^{-1})) = \begin{cases} y & \text{for } i = 2 \\ y^{-1} & \text{for } i = 3 \\ 1 & \text{else.} \end{cases}$$

From this we see that all $[(1, ..., 1, y, 1, ...), \tau]$, i.e. a system of generators of $\mathcal{K}(Y, n)$ lies in the image of p. This proves the surjectivity of $p : X_{\mathfrak{S}_n} \to \mathcal{K}(Y, n)$.

The rest about commutativity of the diagram is straight forward. □

In particular, we can apply this result to $\mathcal{K}(-, n)$ and its $\mathfrak{S}_n$-action. The following result shows that $\mathcal{K}(-, n)$ is in some sense a universal construction:

Corollary 3.10 *If $X = \mathcal{K}(G, n)$ with $n \geq 3$ and φ is the $\mathfrak{S}_n$-action that comes with $\mathcal{K}(G, n)$ then*

$$X_{\mathfrak{S}_n} = X \quad \textit{and} \quad Y \cong G.$$

Moreover, the homomorphism

$$X \quad = \quad X_{\mathfrak{S}_n} \quad \to \quad \mathcal{K}(Y, n)$$

given by Proposition 3.9 is an isomorphism in this case.

PROOF. The fact that $X_{\mathfrak{S}_n} = X$ follows from Lemma 3.1. Using the second short exact sequence of Proposition 3.4 we see that $Y \cong G$.

If we use the fact that

$$\mathcal{K}(G, n)_{\mathfrak{S}_{n-1}^{(i)}} \quad = \quad \mathcal{K}(G, n) \cap (\, G^{i-1} \times \underbrace{\{1\}}_{i\text{.th position}} \times G^{n-i+1} \,) \quad \leq \quad G^n$$

then it is clear that the intersection of all $\mathcal{K}(G, n)_{\mathfrak{S}_{n-1}^{(i)}}$ over $i = 1, ..., n$ is trivial. Hence the homomorphism from $X = X_{\mathfrak{S}_n}$ onto $\mathcal{K}(Y, n)$ is an isomorphism. □

3.4 (Counter-)Examples

We now compute some examples apart from those given by Corollary 3.5. They also provide counter-examples to some naïve ideas the author had about the short exact sequences of Proposition 3.4 and further functoriality properties apart from those given in Proposition 3.8.

We refer to Section 5.6 for further examples and a connection of $\mathcal{E}(-, n)$ with the theory of Coxeter groups.

Example 3.11 *Let $n \geq 2$ be a natural number. Let D_{2k} be the dihedral group of order $2k$. This group is a semidirect product of the cyclic group $\mathbb{Z}_k$ by $\mathbb{Z}_2$.*

1. *If $k \equiv 1 \mod 2$ then $\mathcal{K}(D_{2k}, n) \cong \mathbb{Z}_k \rtimes ({D_{2k}}^{n-1})$.*
2. *If $k \equiv 2 \mod 4$ then $\mathcal{K}(D_{2k}, n) \cong \mathbb{Z}_{k/2} \rtimes ({D_{2k}}^{n-1})$.*
3. *If $k \equiv 0 \mod 4$ then $\mathcal{K}(D_{2k}, n) \cong \left(\mathbb{Z}_{k/2} \times {\mathbb{Z}_k}^{n-1}\right) \rtimes {\mathbb{Z}_2}^{n-1}$.*

We remark that

1. *In the first two cases the exact sequence of Proposition 3.4 splits whereas this sequence is not split for $k \equiv 0 \mod 4$ and $n \geq 3$.*
2. *The subgroup $\mathcal{K}(\mathbb{Z}_k, n)$ of $\mathcal{K}(D_{2k}, n)$ is not normal.*
3. *Even though $\mathcal{K}(D_{2k}, n)$ is a semidirect product it is not not a semidirect product of $\mathcal{K}(\mathbb{Z}_k, n)$ by $\mathcal{K}(\mathbb{Z}_2, n)$.*

PROOF. We will use the presentations

$$D_{2k} = \langle s, d \mid s^2 = d^k = 1, sds = d^{-1} \rangle,\ \mathbb{Z}_k = \langle d \mid d^k = 1 \rangle,\ \mathbb{Z}_2 = \langle s \mid s^2 = 1 \rangle.$$

The commutator subgroup $[D_{2k}, D_{2k}]$ equals $\langle d^2 \rangle$.

If k is odd we then get $[D_{2k}, D_{2k}] = \langle d \rangle \cong \mathbb{Z}_k$. We can split the exact sequence of Proposition 3.4 by sending for $i = 2, ..., n$ the elements $d, s \in D_{2k}$ to

$$\phi_i(d) := (1, ..., 1, d, 1, ..., 1) \quad \text{and} \quad \phi_i(s) := (s, 1, ..., 1, s, 1, ..., 1)$$

(in both cases there is a non trivial entry in the i.th position). After checking that $\phi_i(D_{2k}) \leq \mathcal{K}(D_{2k}, n)$ we see that this splits the projection ${D_{2k}}^n \twoheadrightarrow D_{2k}$ onto the i.th factor. Also it is easy to see that ϕ_i and ϕ_j commute for $i \neq j$. This already proves the assertions in case k is odd.

For k even we have $[D_{2k}, D_{2k}] = \langle d^2 \rangle \cong \mathbb{Z}_{k/2}$. To obtain a splitting of the exact sequence of Proposition 3.4 we have to set for $i = 2, ..., n$

$$\phi_i(s) := (sd^{a_i}, 1, ..., 1, s, 1, ..., 1) \quad \text{and} \quad \phi_i(d) := (d^{b_i}, 1, ..., 1, d, 1, ..., 1)$$

since the product over all components has to lie inside $[D_{2k}, D_{2k}] = \langle d^2\rangle$. For ϕ_i to map into $\mathcal{K}(D_{2k}, n)$ it is necessary that b_i is odd and that a_i is even for all i. For $n \geq 3$ and $i \neq j$ to ensure that $\phi_j(s)$ and $\phi_i(d)$ commute we must have

$$sd^{a_j} \cdot d^{b_i} \;=\; d^{b_i} \cdot sd^{a_j}$$

i.e. $a_j + b_i \equiv a_j - b_i \mod k$. This implies that $2b_i \equiv 0 \mod k$ and so $b_i \equiv 0 \mod (k/2)$. Together with $b_i \equiv 1 \mod 2$ we see that we cannot find a solution for b_i if $k/2$ is even, i.e. if k is divisible by 4. So for $n \geq 3$ and $4|k$ the sequence cannot be split.

If k is even and not divisible by 4 we can set

$$\phi_i(s) := (s, 1, ..., 1, s, 1, ..., 1) \quad \text{and} \quad \phi_i(d) := (d^{k/2}, 1, ..., 1, d, 1, ..., 1)$$

and thus obtain a splitting for $\mathcal{K}(D_{2k}, n)$ in this case. This proves our assertions for $k \equiv 2 \mod 4$.

From Proposition 3.8 we conclude that $\mathcal{K}(D_{2k}, n)$ is a semidirect product of $S := \mathcal{K}(D_{2k}, n) \cap \mathbb{Z}_k{}^n$ (we intersect inside $D_{2k}{}^n$) by $\mathcal{K}(\mathbb{Z}_2, n) \cong \mathbb{Z}_2{}^{n-1}$. It is easy to see that S contains $\mathcal{K}(\mathbb{Z}_k, n)$ and $(..., 1, g, 1, ...)$ with $g \in [D_{2k}, D_{2k}]$. But $(..., 1, d, 1, ...)$ cannot lie in S because it is not even an element of $\mathcal{K}(D_{2k}, n)$. This implies that S is generated inside $\mathbb{Z}_k{}^n$ by the elements $(..., 1, d^2, 1, ...)$ and $\mathcal{K}(\mathbb{Z}_k, n)$. This is enough to identify S as $\mathbb{Z}_{k/2} \times \mathbb{Z}_k{}^{n-1}$. □

Example 3.12 *Let Q_8 be the quaternion group and $D_8 = D_{2\cdot 4}$ the dihedral group of order 8. Then there are isomorphisms*

$$\mathcal{K}(Q_8,\, 2) \;\cong\; \mathbb{Z}_2 \times Q_8 \qquad \textit{and} \qquad \mathcal{K}(D_8,\, 2) \;\cong\; \mathbb{Z}_2 \times D_8.$$

In particular, we have $\mathcal{K}(G^{\mathrm{ab}},\, 2) \not\cong \mathcal{K}(G,\, 2)^{\mathrm{ab}}$ for G equal to D_8 or Q_8.

PROOF. We will use the presentation

$$Q_8 \;=\; \langle a, b \,|\, a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1}\rangle.$$

The commutator group $[Q_8, Q_8]$ equals $\langle a^2\rangle$. So we get $Q_8{}^{\mathrm{ab}} \cong \mathbb{Z}_2{}^2$ and therefore $\mathcal{K}(Q_8{}^{\mathrm{ab}}, 2) \cong \mathbb{Z}_2{}^2$.

We see that $\mathcal{K}(Q_8, 2)$ is generated inside $Q_8{}^2$ by the elements $x := (a^{-1}, a)$, $y := (b^{-1}, b)$ (giving a set theoretical section of Q_8 to $\mathcal{K}(Q_8, 2)$) and $z := (a^2, 1)$ (being a generator for the kernel of the surjection $\mathcal{K}(Q_8, 2) \twoheadrightarrow Q_8$). It is easy to see that $\langle x, y\rangle \cong Q_8$ and that $\langle z\rangle$ commutes with $\langle x, y\rangle$. Now $\mathcal{K}(Q_8, 2)$ has order 16 being an extension of $[Q_8, Q_8] \cong \mathbb{Z}_2$ by Q_8. Therefore there are no further relations among the x, y, z and there exists an isomorphism $\mathcal{K}(Q_8, 2) \cong \mathbb{Z}_2 \times Q_8$. In particular, $\mathcal{K}(Q_8, 2)^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}_2{}^3$.

The proof for D_8 is similar to the case of Q_8 and is left to the reader. □

Example 3.13 *Let Q_8 be the quaternion group. We denote by $Z(Q_8)$ the centre of Q_8. Then the short exact sequence*

$$1 \;\rightarrow\; Z(Q_8) \;\rightarrow\; Q_8 \;\rightarrow\; \mathbb{Z}_2 \times \mathbb{Z}_2 \;\rightarrow\; 1$$

induces for all $n \geq 3$ a sequence

$$1 \;\rightarrow\; \mathcal{K}(Z(Q_8),\, n) \;\rightarrow\; \mathcal{K}(Q_8,\, n) \;\rightarrow\; \mathcal{K}(\mathbb{Z}_2 \times \mathbb{Z}_2,\, n) \;\rightarrow\; 1$$

that is not exact in the middle.

We remark that $\mathcal{K}(Z(Q_8),\, n)$ is a normal subgroup of $\mathcal{K}(Q_8,\, n)$.

In particular, also the subgroup of $\mathcal{K}(Q_8,\, n)$ generated by the conjugates of $\mathcal{K}(Z(Q_8),\, n)$ does not give the kernel of the surjective homomorphism from $\mathcal{K}(Q_8,\, n)$ onto $\mathcal{K}(\mathbb{Z}_2 \times \mathbb{Z}_2,\, n)$.

PROOF. Using the presentation of Q_8 as in Example 3.12 we have $Z(Q_8) = \langle a^2 \rangle = \langle b^2 \rangle$. We can identify $\mathcal{K}(Z(Q_8), n)$ with the subgroup of $Z(Q_8)^n$ where the product over all components equals 1.

Since $Z(Q_8)$ is the centre of Q_8 we see that $\mathcal{K}(Z(Q_8), n)$ is a normal subgroup of ${Q_8}^n$ and hence also a normal subgroup of $\mathcal{K}(Q_8, n)$.

The kernel of the surjective homomorphism $\mathcal{K}(Q_8, n) \twoheadrightarrow \mathcal{K}(\mathbb{Z}_2 \times \mathbb{Z}_2, n)$ equals $\mathcal{K}(Q_8, n) \cap Z(Q_8)^n$. However, the element $(a^2, 1, ..., 1)$ lies in this kernel but not in $\mathcal{K}(Z(Q_8), n)$. □

4 A first quotient of $\pi_1(X_{\mathrm{gal}})$ and $\pi_1(X_{\mathrm{gal}}^{\mathrm{aff}})$

Das Firmament blaut ewig, und die Erde
Wird lange fest steh'n und aufblühn im Lenz.
Du aber, Mensch, wie lang lebst denn du?
Nicht hundert Jahre darfst du dich ergötzen
An all dem morschen Tande dieser Erde!

4.1 Étale and topological fundamental groups

In this section we recall some well-known facts that can be found e.g. in [SGA1, Exposé XII].

Let X be an irreducible normal scheme of finite type over the complex numbers and let X^{an} be its associated complex analytic space. Then we consider the following three categories:

1. The objects are connected and finite étale covers $Y \to X$ where Y is an algebraic scheme and the morphisms are morphisms of schemes over X between these covers.

2. The objects are connected holomorphic covers $\mathfrak{Y} \to X^{\mathrm{an}}$ where $\mathfrak{Y}$ is a complex space and the morphisms are holomorphic morphisms of complex spaces over X^{an} between these covers.

3. The objects are connected topological covers $\mathcal{Y} \to X^{\mathrm{an}}$ where $\mathcal{Y}$ is a topological space and the morphisms are continuous maps of topological spaces over X^{an} between these covers.

The relationship between these three categories is as follows:

- Given a finite étale cover $p : Y \to X$ by a scheme Y this induces a finite holomorphic cover $p^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$.

  Moreover, every algebraic morphism between finite étale covers of X induces a unique holomorphic morphism between their analytifications.

- Every holomorphic cover is also a topological cover and every holomorphic map is continuous.

- Every topological cover of X^{an} can be given a unique structure of a complex space such that the projection map onto X^{an} becomes holomorphic.

  Moreover, every continuous map between holomorphic covers over X^{an} can be given a unique structure of a holomorphic morphism.

- By Riemann's existence theorem every finite holomorphic cover $\mathfrak{Y} \to X^{\mathrm{an}}$ is algebraic, i.e. there exists a finite étale cover $p : Y \to X$ by an algebraic scheme Y such that the associated analytification is isomorphic to $\mathfrak{Y} \to X^{\mathrm{an}}$.

This implies that there is an equivalence of categories between the category of holomorphic covers of X^{an} and the category of topological covers of X^{an}. Both categories can be described in terms of discrete sets with an action of the fundamental group $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, x_0)$ on it where x_0 is a point of X^{an}.

As explained in [SGA1] there exists a profinite group $\pi_1^{\text{ét}}(X, x_0)$ that classifies connected and finite étale covers of X.

By the above the category of finite étale covers of the scheme X is equivalent to the full subcategory of *finite* holomorphic covers of the category of holomorphic covers of X^{an}. So there is a natural homomorphism $\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, x_0) \to \pi_1^{\text{ét}}(X, x_0)$ that identifies the finite quotients of both groups. Hence this map induces an isomorphism

$$\widehat{\pi_1^{\mathrm{top}}}(X^{\mathrm{an}},\, x_0) \;\cong\; \pi_1^{\text{ét}}(X,\, x_0)$$

where $\widehat{\ }$ denotes the profinite completion of a group.

The homomorphism of a group to its profinite completion is in general not surjective as the example $\mathbb{Z} \to \widehat{\mathbb{Z}}$ shows. However, the image of a group inside its profinite completion is always dense with respect to the profinite topology. We recall that a group G is called *residually finite* if the natural homomorphism from G to its profinite completion $\widehat{G}$ is injective. There do exist finitely presented groups that are not residually finite, e.g. Higman's 4-group [Se, Chapter I.1.4].

Serre asked in loc. cit. whether there are complex algebraic varieties that have non-residually finite fundamental groups. The facts are as follows:

1. If X is a smooth complex projective algebraic curve then $\pi_1^{\mathrm{top}}(X, x_0)$ being a subgroup of $SL_2(\mathbb{R})$ is residually finite (cf. [LS, Proposition III.7.11] and Section 1.1).

2. If X is a smooth complex affine algebraic curve then $\pi_1^{\mathrm{top}}(X, x_0)$ is a free group and hence residually finite.

3. Toledo [To] constructed smooth complex projective algebraic surfaces with fundamental groups that are not residually finite.

So having proven Theorem 4.3 there is no way to deduce from it the corresponding statement for topological fundamental groups since we are dealing with algebraic surfaces and so the fundamental groups involved may not be residually finite.

4.2 The quotient for the étale fundamental group

For the following quite elementary treatment of étale fundamental groups in terms of Galois groups and the arguments on inertia groups used we refer the reader to the book of H. Popp [Popp] for proofs and details.

We let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n with Galois closure $f_{\text{gal}} : X_{\text{gal}} \to \mathbb{P}^2$. We define the following function fields:

$$\begin{array}{lcll} L & := & \text{function field of} & X_{\text{gal}} \\ K & := & \text{function field of} & X \\ k & := & \text{function field of} & \mathbb{P}^2. \end{array}$$

We will assume that they are all contained in a fixed algebraically closed field Ω. We have already seen in Proposition 2.7 that the Galois group of L/k is isomorphic to the symmetric group $\mathfrak{S}_n$. We may assume that K is the fixed field of $\mathfrak{S}_{n-1}^{(1)}$ in the notation of Definition 2.10.

We denote by K^{nr} the maximal unramified extension of K i.e. the compositum of all finite field extensions inside Ω of K such that the normalisation of X in these fields is étale over X. We similarly denote by L^{nr} the maximal unramified extension of L and by k^{nr} the maximal unramified extension of k. Of course, we have $k^{\text{nr}} = k$. But for later generalisations it is better to use this fact as late as possible.

More or less by definition of the étale fundamental groups there are isomorphisms of profinite groups

$$\pi_1^{\text{ét}}(X_{\text{gal}}) \;\cong\; \text{Gal}(L^{\text{nr}}/L) \quad \text{and} \quad \pi_1^{\text{ét}}(X) \;\cong\; \text{Gal}(K^{\text{nr}}/K).$$

To be more precise, there is an isomorphism of $\pi_1^{\text{ét}}(X, \text{Spec}\,\Omega)$ with the opposite group $\text{Gal}(K^{\text{nr}}/K)$ that depends on the choice of the embedding of K into Ω. Of course there are similar dependencies for $\pi_1^{\text{ét}}(X_{\text{gal}})$ and $\pi_1^{\text{ét}}(\mathbb{P}^2)$. So we fix $\text{Spec}\,\Omega$ as base point for all étale fundamental groups occurring in this section. Since we fixed Ω and embeddings of the fields k, K, L, k^{nr}, K^{nr} and L^{nr} into Ω we will not mention base points and identify the étale fundamental groups with their corresponding Galois groups with these choices understood. We refer to Section 4.3 for more details on these choices.

Both extensions L/k and L^{nr}/L are Galois. It is easy to see that L^{nr}/k also is a Galois extension: The Galois closure of L^{nr}/k would have to be unramified over L i.e. must be contained in L^{nr}. Hence there is a short exact sequence

$$1 \;\to\; \text{Gal}(L^{\text{nr}}/L) \;\to\; \text{Gal}(L^{\text{nr}}/k) \;\to\; \text{Gal}(L/k) \;\to\; 1$$

with $\text{Gal}(L/k) \cong \mathfrak{S}_n$.

We let Y be a connected and finite étale cover of X_{gal} with function field M that we will assume to be Galois over k and to be a subfield of Ω. We let R be the coordinate ring of a generic open affine subset of $\mathbb{P}^2$, so that in particular k is the field of fractions of R. We let S be the integral closure of R inside L and let T be the integral closure of R inside M. Since $f : X \to \mathbb{P}^2$ is a good generic projection the branch locus D is an irreducible curve inside $\mathbb{P}^2$ and hence corresponds to a prime ideal $\mathfrak{p}$ of height 1 of R. From Proposition 2.12 we see that $\mathfrak{p}$ splits in S into a product

$$\mathfrak{p} \;=\; \prod_{\tau} \mathfrak{P}_{\tau}^2$$

where τ runs through the transpositions of $\mathfrak{S}_n$ and the $\mathfrak{P}_\tau$'s correspond to the irreducible curves R_τ as defined in Section 2.3. Since T is étale over S the $\mathfrak{P}_\tau$'s do not ramify in T. Each $\mathfrak{P}_\tau$ splits into a product of $\mathfrak{Q}_{\tau,i}$'s $i = 1, ..., \ell$ where ℓ divides the degree of the extension M/L. We thus get the following picture:

varieties:	$\mathbb{P}^2$	$\leftarrow$	X_{gal}	$\leftarrow$	Y
function fields:	k	$\subseteq$	L	$\subseteq$	M
coordinate rings:	R	$\subseteq$	S	$\subseteq$	T
prime ideals:	$\mathfrak{p}$	$=$	$\prod_\tau \mathfrak{P}_\tau^2$	$=$	$\prod_{i=1}^{\ell} \prod_\tau \mathfrak{Q}_{\tau,i}^2$

Since the ramification indices $e(\mathfrak{P}_\tau/\mathfrak{p}) = e(\mathfrak{Q}_{\tau,i}/\mathfrak{p})$ are all equal to 2, we conclude that the inertia groups are subgroups of $\mathrm{Gal}(M/k)$ isomorphic to $\mathbb{Z}_2$. Under the natural homomorphism $\mathrm{Gal}(M/k) \twoheadrightarrow \mathfrak{S}_n$ the non-trivial element of the inertia group of $\mathfrak{Q}_{\tau,i}$ maps to τ.

Galois theory provides us with the following two short exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Gal}(M/L) & \to & \mathrm{Gal}(M/k) & \to & \mathrm{Gal}(L/k) & \to & 1 \\ & & || & & \uparrow & & \uparrow & & \\ 1 & \to & \mathrm{Gal}(M/L) & \to & \mathrm{Gal}(M/K) & \to & \mathrm{Gal}(L/K) & \to & 1 \end{array}$$

The arrows upwards are injective. We identify $\mathrm{Gal}(L/k)$ with $\mathfrak{S}_n$ and $\mathrm{Gal}(L/K)$ with $\mathfrak{S}_{n-1}^{(1)}$.

We let $N^{(1)}$ be the subgroup of $\mathrm{Gal}(M/K)$ normally generated (with respect to $\mathrm{Gal}(M/K)$) by the inertia groups of all prime ideals $\mathfrak{Q}_{\tau,i}$ lying above prime ideals $\mathfrak{P}_\tau$ with $\tau \in \mathfrak{S}_{n-1}^{(1)}$:

$$N^{(1)} \;=\; \ll \mathrm{I}(\mathfrak{Q}_{\tau,i}) \,|\, \forall i, \forall \tau \in \mathfrak{S}_{n-1}^{(1)} \gg \;\trianglelefteq\; \mathrm{Gal}(M/K).$$

If we view $N^{(1)}$ as a subgroup of $\mathrm{Gal}(M/k)$ then it maps to $\mathfrak{S}_{n-1}^{(1)}$ under the homomorphism onto $\mathrm{Gal}(L/k)$. Hence the fixed field $\mathrm{Fix}(N^{(1)})$ is a Galois extension of K. The normalisation of X inside $\mathrm{Fix}(N^{(1)})$ is a finite étale cover since we quotiened out all inertia groups. Hence it is contained in the field $M \cap K^{\mathrm{nr}}$ where we intersect these fields inside Ω. Conversely, there is a surjective homomorphism from $\mathrm{Gal}(M/K)$ onto $\mathrm{Gal}(M \cap K^{\mathrm{nr}}/K)$. Since the inertia groups cannot survive under this surjection we conclude that $N^{(1)}$ must be contained in the kernel of this homomorphism. Putting this together we see that there is a short exact sequence

$$1 \;\to\; N^{(1)} \;\to\; \mathrm{Gal}(M/K) \;\to\; \mathrm{Gal}(M \cap K^{\mathrm{nr}}/K) \;\to\; 1.$$

In a similar fashion we define N to be the subgroup of $\mathrm{Gal}(M/k)$ that is normally generated by the inertia groups of all the $\mathfrak{Q}_{\tau,i}$. With the same arguments as above we conclude that N generates the kernel of the surjective homomorphism from $\mathrm{Gal}(M/k)$ onto $\mathrm{Gal}(M \cap k^{\mathrm{nr}}/k)$.

For the non-trivial elements r_1 and r_2 of two inertia groups of $\mathfrak{Q}_{\tau_1,i_1}$ and $\mathfrak{Q}_{\tau_2,i_2}$ we define (cf. Definition 2.11)

$$c(r_1,\, r_2) := \begin{cases} 1 & \text{if } \tau_1 = \tau_2 \\ r_1 r_2 r_1^{-1} {r_2}^{-1} & \text{if } \tau_1 \text{ and } \tau_2 \text{ are disjoint} \\ r_1 r_2 r_1 {r_2}^{-1} {r_1}^{-1} {r_2}^{-1} & \text{if } \tau_1 \text{ and } \tau_2 \text{ are cuspidal.} \end{cases}$$

We define C to be subgroup normally generated inside $\mathrm{Gal}(M/k)$ by all the $c(r_1, r_2)$'s where the r_i's run through all inertia groups of height one prime ideals inside $\mathrm{Gal}(M/k)$.

Lemma 4.1 *The subgroup C is contained in* $\mathrm{Gal}(M/L)$*, N and $N^{(1)}$.*

PROOF. All $c(r_1, r_2)$'s lie in $\mathrm{Gal}(M/L)$ and N and so the first two inclusions are clear.

We let τ_1 and τ_2 be two disjoint transpositions and choose two inertia elements r_1 and r_2 of $\mathrm{Gal}(M/k)$ mapping to τ_1 and τ_2, respectively. One of the τ_i's, say τ_1, lies in $\mathfrak{S}_{n-1}^{(1)}$. Hence r_1 lies in $N^{(1)}$. Suppose r_2 is the inertia element of some prime ideal $\mathfrak{Q}_{\tau_2,j}$. Then the element $r_2 r_1 {r_2}^{-1}$ is an element of the inertia group of the prime ideal $r_2 \mathfrak{Q}_{\tau_2,j}$. This latter inertia group is contained in $N^{(1)}$ and so $r_2 r_1 {r_2}^{-1}$ lies in $N^{(1)}$. Hence $c(r_1, r_2)$ lies in $N^{(1)}$.

We leave the case of two cuspidal transpositions to the reader. □

Hence the maps from $\mathrm{Gal}(M/k)$ onto $\mathrm{Gal}(M \cap k^{\mathrm{nr}}/k)$ and from $\mathrm{Gal}(M/K)$ onto $\mathrm{Gal}(M \cap K^{\mathrm{nr}}/K)$ factor over the quotient by C. And so we obtain the following two short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Gal}(M/L) & \to & \mathrm{Gal}(M/k) & \to & \mathrm{Gal}(L/k) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathrm{Gal}(M/L)/C & \to & \mathrm{Gal}(M/k)/C & \to & \mathrm{Gal}(L/k) & \to & 1 \quad (*) \end{array}$$

where the arrows downwards are surjective.

Proposition 4.2 *We can split the short exact sequence* $(*)$ *using inertia groups. With respect to this splitting there are the following isomorphisms*

$$\begin{array}{rcl}(\mathrm{Gal}(M/L)/C) \;/\; (\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n} & \cong & \mathrm{Gal}(M\cap k^{\mathrm{nr}}/k)\\ (\mathrm{Gal}(M/L)/C) \;/\; (\mathrm{Gal}(M/L)/C)_{\mathfrak{S}^{(1)}_{n-1}} & \cong & \mathrm{Gal}(M\cap K^{\mathrm{nr}}/K)\end{array}$$

where the notations are the ones introduced in Section 3.1

If Question 2.14 has an affirmative answer for the finite étale cover $Y \to X_{\mathrm{gal}}$ *then the group* C *is trivial.*

PROOF. For every transposition $(1\,k)$ we choose a prime ideal $\mathfrak{Q}_{(1\,k),i}$ and denote by r_k the non-trivial element of its inertia group. We denote by $\bar{r}_k$ the image of r_k inside $\mathrm{Gal}(M/k)/C$. The elements $\bar{r}_k$ fulfill $\bar{r}_k{}^2 = 1$ and map to $(1\,k)$ under the induced surjection onto $\mathrm{Gal}(L/k)$. Since we took the quotient by C also the following relations hold true:

$$(\bar{r}_i\,\bar{r}_{i+1})^3 \;=\; 1 \text{ and } (\bar{r}_i\,\bar{r}_j)^2 \;=\; 1 \text{ for } |i-j|\geq 2.$$

These are precisely the Coxeter relations for $\mathfrak{S}_n$ (cf. Section 5.6) and hence the $\bar{r}_k$ define a group isomorphic to a quotient of $\mathfrak{S}_n$. Since there is a surjective map from this group onto $\mathfrak{S}_n$ it must be equal to $\mathfrak{S}_n$. This defines a splitting $s : \mathrm{Gal}(L/k) \to \mathrm{Gal}(M/k)/C$.

From Lemma 4.1 we know that C is a subgroup of N. So we see that the map from $\mathrm{Gal}(M/k)$ onto $\mathrm{Gal}(M\cap k^{\mathrm{nr}}/k)$ factors over $\mathrm{Gal}(M/k)/C$. The kernel of the map from $\mathrm{Gal}(M/k)/C$ onto $\mathrm{Gal}(M\cap k^{\mathrm{nr}}/k)$ clearly is the image $\bar{N}$ of N inside $\mathrm{Gal}(M/k)/C$. The group $\bar{N}$ is generated by the images of the inertia groups.

From Lemma 3.1 we know that $(\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n}$ is generated by the commutators $[g, s(\tau)]$'s where g runs through $\mathrm{Gal}(M/L)$ and τ runs through the transpositions of $\mathfrak{S}_n$. The element $gs(\tau)g^{-1}$ is the non-trivial element of the inertia group of some prime ideal lying above $\mathfrak{P}_\tau$. With this said it is easy to conclude the equalities

$$\begin{array}{rrcl} & N & = & (\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n}\cdot s(\mathfrak{S}_n)\\ \text{and} & N\cap(\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n} & = & (\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n}\end{array}$$

Applying the second isomorphism theorem of group theory we obtain

$$\begin{array}{rcccl}\dfrac{\mathrm{Gal}(M/L)/C}{(\mathrm{Gal}(M/L)/C)_{\mathfrak{S}_n}} & = & \dfrac{\mathrm{Gal}(M/L)/C}{N\cap \mathrm{Gal}(M/L)/C} & = & \dfrac{N\cdot \mathrm{Gal}(M/L)/C}{N}\\ & = & \dfrac{\mathrm{Gal}(M/k)/C}{N} & \cong & \mathrm{Gal}(M\cap k^{\mathrm{nr}}/k).\end{array}$$

Hence the induced homomorphism from $\mathrm{Gal}(L/K)/C$ to $\mathrm{Gal}(M \cap k^{\mathrm{nr}}/k)$ is surjective with kernel $(\mathrm{Gal}(L/K)/C)_{\mathfrak{S}_n}$.

The assertion about the quotient of $\mathrm{Gal}(M/L)/C$ by $(\mathrm{Gal}(M/L)/C)_{\mathfrak{S}^{(1)}_{n-1}}$ is proved similarly and left to the reader.

Now suppose that the curves corresponding to the $\mathfrak{Q}_{\tau,i}$'s fulfill the connectivity properties of Question 2.14.

For two disjoint transpositions τ_1 and τ_2 we choose two prime ideals $\mathfrak{Q}_{\tau_1,i}$ and $\mathfrak{Q}_{\tau_2,j}$ and let r_1 and r_2 be the non-trivial elements of their inertia groups. Since the curves corresponding to the two prime ideals intersect there is a maximal ideal containing both of them. The inertia group of this maximal ideal is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and is generated by r_1 and r_2. Hence these two elements commute and $c(r_1, r_2) = 1$.

If τ_1 and τ_2 have exactly one index in common then there is a maximal ideal with inertia group $\mathfrak{S}_3$ that is generated by r_1 and r_2. So there is a triple commutator relation between r_1 and r_2 and so also $c(r_1, r_2) = 1$ holds true in this case.

So if Question 2.14 has an affirmative answer for $Y \to X_{\mathrm{gal}}$ then all the $c(r_1, r_2)$'s are equal to 1 and so C is trivial. $\square$

We now pass to the limit of all finite étale covers of X_{gal} and keep track of the induced homomorphisms between the corresponding field extensions and their Galois groups. We will denote the limit of the subgroups C by C^{proj}. Using Proposition 4.2 we arrive at surjective homomorphisms

$$\begin{array}{rcl} \mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}} & \twoheadrightarrow & \mathrm{Gal}(L^{\mathrm{nr}} \cap k^{\mathrm{nr}}/k) \\ \mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}} & \twoheadrightarrow & \mathrm{Gal}(L^{\mathrm{nr}} \cap K^{\mathrm{nr}}/K). \end{array}$$

Taking the compositum of L with K^{nr} we get a subfield of Ω that corresponds to a limit of étale extensions X_{gal}. Hence this compositum must be contained in L^{nr} and hence already K^{nr} was contained in L^{nr}. So $K^{\mathrm{nr}} \cap L^{\mathrm{nr}}$ is equal to K^{nr} and the second surjective homomorphism above takes the form

$$\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}} \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{nr}}/K).$$

Its kernel is $(\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}})_{\mathfrak{S}^{(1)}_{n-1}}$.

Up to now have actually never needed that k is the function field of the projective plane over the complex numbers. This means that everything done in this section works equally well in the affine situation. We denote by $L^{\mathrm{nr,aff}}$ the compositum of all fields corresponding to finite étale extensions of $X^{\mathrm{aff}}_{\mathrm{gal}}$ inside Ω. We then define C^{aff} to be the subgroup of $\mathrm{Gal}(L^{\mathrm{nr,aff}}/k)$ normally generated by the $c(r_1, r_2)$'s where the r_i's run through inertia groups in this extension.

No matter whether we are in the affine or the projective situation, since k is the function field of the affine or the projective plane over the complex numbers there are no non-trivial étale covers and so $k^{\mathrm{nr}} = k$ as we already mentioned much earlier. In particular, $\mathrm{Gal}(L^{\mathrm{nr}} \cap k^{\mathrm{nr}}/k)$ is trivial and so

$$\begin{array}{rcl} (\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}})_{\mathfrak{S}_n} & = & \mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}} \\ (\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}})_{\mathfrak{S}_{n-1}^{(1)}} & = & \ker(\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}} \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{nr}}/K)) \end{array}$$

where we have shown the second equality already above. Of course, we also get the corresponding statements for the affine situation. Applying Proposition 3.9 to $\mathrm{Gal}(L^{\mathrm{nr}}/L)/C^{\mathrm{proj}}$ and $\mathrm{Gal}(L^{\mathrm{nr,aff}}/L)/C^{\mathrm{aff}}$ we obtain

Theorem 4.3 *Let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n with Galois closure X_{gal}. Then there are surjective homomorphisms*

$$\begin{array}{rclcl} \pi_1^{\text{ét}}(X_{\mathrm{gal}}) & \twoheadrightarrow & \pi_1^{\text{ét}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \twoheadrightarrow & \mathcal{K}(\pi_1^{\text{ét}}(X),\, n) \\ \pi_1^{\text{ét}}(X_{\mathrm{gal}}^{\mathrm{aff}}) & \twoheadrightarrow & \pi_1^{\text{ét}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \twoheadrightarrow & \mathcal{K}(\pi_1^{\text{ét}}(X^{\mathrm{aff}}),\, n). \end{array}$$

If Question 2.14 has an affirmative answer for all finite étale covers of $X_{\mathrm{gal}}^{\mathrm{aff}}$ then both C^{aff} and C^{proj} are trivial.

If Question 2.14 has an affirmative answer for all finite étale covers of X_{gal} then at least C^{proj} is trivial.

Even if C^{aff} is trivial we cannot expect these surjective homomorphisms to be isomorphisms. We refer to Theorem 6.2 for details.

The quotient in positive characteristic

Only for the rest of this section we let X be a smooth projective surface over an arbitrary algebraically closed field of characteristic $\neq 2, 3$.

For every finite and separable morphism we can form its Galois closure. We say that a finite separable morphism $f : X \to \mathbb{P}^2$ is a good generic projection if it fulfills the conditions of a generic projection and if the conclusions of Proposition 2.7 and Proposition 2.12 hold true.

Then nearly all arguments given in Section 4.2 also work in this situation. We assumed that the characteristic of the ground field is $\neq 2, 3$ and since all inertia groups occurring are $\mathbb{Z}_2$, ${\mathbb{Z}_2}^2$ and $\mathfrak{S}_3$ there are no problems with wild ramification. But nearly at the end we used the fact that the affine and the projective plane over the complex numbers are algebraically simply connected.

This is still true for the projective plane over an arbitrary algebraically closed field and for the affine plane over an algebraically closed field of characteristic zero. Hence the proof also works always in the projective case and always in the affine case if we work in characteristic zero.

However, the affine plane is not algebraically simply connected in positive characteristic. In this case our result is still true for the prime-to-p part of the étale fundamental groups in question. The general picture in the affine case is as follows: For an irreducible and affine scheme A together with a finite morphism $a : A \to \mathbb{A}^2$ we define the *new part* of the étale fundamental group of A to be the kernel

$$\pi_1^{\acute{e}t,new}(a,\, A) \;\; := \;\; \ker(a_* \,:\, \pi_1^{\text{ét}}(A) \;\to\; \pi_1^{\text{ét}}(\mathbb{A}^2)).$$

This name is motivated by the fact that in our case $f_* : \pi_1^{\text{ét}}(X^{\text{aff}}) \to \pi_1^{\text{ét}}(\mathbb{A}^2)$ and $f_{\text{gal},*} : \pi_1^{\text{ét}}(X_{\text{gal}}^{\text{aff}}) \to \pi_1^{\text{ét}}(\mathbb{A}^2)$ are surjective homomorphisms and since $\pi_1^{\text{ét}}(\mathbb{A}^2)$ is highly non-trivial in positive characteristic we are only interested in the “new part“ coming from the morphisms f and f_{gal}.

We leave it to the reader to use Theorem 3.9 together with the results of Section 4.2 to obtain a surjective homomorphism

$$\pi_1^{\acute{e}t,new}(f_{\text{gal}},\, X_{\text{gal}}^{\text{aff}}) \;\; \twoheadrightarrow \;\; \mathcal{K}(\pi_1^{\acute{e}t,new}(f,\, X^{\text{aff}}),\, n).$$

This is a sort of relative version of Theorem 4.3 that does not involve knowing the group $\pi_1^{\text{ét}}(\mathbb{A}^2)$.

4.3 Classifying covers with group actions

In the following we recall some basic facts on fundamental groups from the point of view of Galois categories and fibre functors. The standard reference is [SGA1]. We especially refer the reader to [SGA1, Exposé V]. The category of G-covers is introduced in [SGA1, Remarque IX.5.8]. For the topological details we refer e.g. to [Di, Kapitel I.9].

Let $\mathfrak{X}$ be a normal irreducible complex analytic space and G a finite group of automorphisms of $\mathfrak{X}$ and so acting from the left on this space. We define the following two categories

$\mathcal{C}$ Covers of $\mathfrak{X}$

The objects are holomorphic covers $\mathfrak{Y} \to \mathfrak{X}$.

The morphisms are holomorphic maps between these covers over $\mathfrak{X}$.

$\mathcal{C}_{\text{G}}$ G-Covers of $\mathfrak{X}$

The objects are holomorphic covers $p : \mathfrak{Y} \to \mathfrak{X}$ together with a left G-action on $\mathfrak{Y}$ that is compatible with the G-action on $\mathfrak{X}$ via p.

The morphisms are G-equivariant holomorphic maps between these covers over $\mathfrak{X}$.

We already said in Section 4.1 that $\mathcal{C}$ is equivalent to the category of topological covers of $\mathfrak{X}$. We identify $\mathcal{C}$ with $\mathcal{C}_{\{1\}}$ where $\{1\}$ denotes the trivial group.

We recall that an object Y of a category in which coproducts exist is called *connected* if it is not isomorphic to a coproduct $Y_1 \coprod Y_2$ where Y_1 and Y_2 are objects of this category not isomorphic to the initial object.

For an arbitrary discrete group π_1 we define the following category:

$\mathcal{C}(\pi_1)$ π_1-sets

The objects are discrete sets with a left or right action of the group π_1.

The morphisms are π_1-equivariant maps between these sets.

We warn the reader that when discussing the fundamental group π_1 in algebraic geometry one often considers sets with *left* π_1-actions whereas in topology one usually considers sets with *right* π_1-actions. Therefore the author decided to be rather pedantic about this point, especially after he was trapped when he was not paying attention to it.

We choose a universal cover $\tilde{p} : \tilde{\mathfrak{X}} \to \mathfrak{X}$ of $\mathfrak{X}$ in the sense of topology. We denote by $\mathrm{Aut}(\tilde{\mathfrak{X}})$ the group of deck transformations of $\tilde{\mathfrak{X}}$ over $\mathfrak{X}$. Then we define $\pi_1^{\mathrm{top}}(\mathfrak{X}, \tilde{\mathfrak{X}})$ to be the opposite group to $\mathrm{Aut}(\tilde{\mathfrak{X}})$. Given a (connected) cover $p : \mathfrak{Y} \to \mathfrak{X}$ the group $\mathrm{Aut}(\tilde{\mathfrak{X}})$ acts from the right on the set of cover morphisms $\mathrm{Hom}(\tilde{\mathfrak{X}}, \mathfrak{Y})$. Hence there is a left $\pi_1^{\mathrm{top}}(\mathfrak{X}, \tilde{\mathfrak{X}})$-action on this set. This defines a *fibre functor* from the category $\mathcal{C}$ to the category of sets with a left $\pi_1^{\mathrm{top}}(\mathfrak{X}, \tilde{\mathfrak{X}})$-action and makes $\mathcal{C}$ into a *Galois category*.

Here we have to relax Grothendieck's terminology a little bit: We also allow quotients by discrete groups rather than only finite ones. Also we assume that the fibre functor maps to the category of discrete sets with a group action of a discrete group rather than only to the category of finite sets together with a continuous action of a profinite group.

Conversely, given a fibre functor F there is always a group $\pi_1^{\mathrm{top}}(\mathfrak{X}, F)$ called the *automorphism group* of the functor F. A map between two covers $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ over $\mathfrak{X}$ is uniquely determined by the $\pi_1^{\mathrm{top}}(\mathfrak{X}, F)$-equivariant map from $F(\mathfrak{Y}_1)$ to $F(\mathfrak{Y}_2)$. The main content of Galois theory and the theory of the fundamental group in this setup is that a fibre functor induces an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}(\pi_1)$.

The connection with the fundamental group defined via loops is as follows: We let $F_{x_0}(\mathfrak{Y}) := p^{-1}(x_0)$ be the fibre of $p : \mathfrak{Y} \to \mathfrak{X}$ above a point x_0 of $\mathfrak{X}$. Lifting loops based at x_0 to paths in $\mathfrak{Y}$ defines a right action of the "loop"-fundamental group $\pi_1^{\mathrm{top}}(\mathfrak{X}, x_0)$ on the set $F_{x_0}(\mathfrak{Y})$. Now we fix a point $\tilde{x}_0$ on the fibre $F_{x_0}(\tilde{\mathfrak{X}})$ of a universal cover. Then we compare the left $\mathrm{Aut}(\tilde{\mathfrak{X}})$-action with the right $\pi_1^{\mathrm{top}}(\mathfrak{X}, x_0)$-action in this point: For every automorphism φ there is a unique element γ in the fundamental group such that $\varphi \cdot \tilde{x}_0 = \tilde{x}_0 \cdot \gamma$. This defines an isomorphism between these two groups that depends on the choice of $\tilde{x}_0$. In fact, given another point $\tilde{x}_0'$ of $F_{x_0}(\tilde{\mathfrak{X}})$ there is a unique element $\alpha \in \pi_1^{\text{top}}(\mathfrak{X}, x_0)$ such that $\tilde{x}_0' = \tilde{x}_0 \cdot \alpha$. We then compute

$$\varphi \cdot \tilde{x}_0' = \varphi \cdot (\tilde{x}_0 \cdot \alpha) = (\varphi \cdot \tilde{x}_0) \cdot \alpha = (\tilde{x}_0 \cdot \gamma) \cdot \alpha = \tilde{x}_0' \cdot (\alpha^{-1}\gamma\alpha).$$

That is, with respect to $\tilde{x}_0'$ we obtain an isomorphism of $\pi_1^{\text{top}}(\mathfrak{X}, x_0)$ with $\mathrm{Aut}(\tilde{\mathfrak{X}})$ that differs from the isomorphism with respect to $\tilde{x}_0$ by conjugation with α.

If we fix a point $\tilde{x}_0$ of $\tilde{\mathfrak{X}}$ in the fibre $\tilde{p}^{-1}(x_0)$ we can identify $\mathrm{Hom}(\tilde{\mathfrak{X}}, \mathfrak{Y})$ with the set $F_{x_0}(\mathfrak{Y})$ by associating to a morphism $\varphi : \tilde{\mathfrak{X}} \to \mathfrak{Y}$ the point $\varphi(\tilde{x}_0)$. Under this identification the right action of $\pi_1^{\text{top}}(\mathfrak{X}, x_0)$ on $F_{x_0}(\mathfrak{Y})$ becomes a left action on $\mathrm{Hom}(\tilde{\mathfrak{X}}, \mathfrak{Y})$ and it is this point where the group actions change their side when passing from topology to algebraic geometry and vice versa.

For a cover $p : \mathfrak{Y} \to \mathfrak{X}$ we recover the group of its automorphisms as follows: The group $\mathrm{Aut}(\tilde{\mathfrak{X}})$ acts on $\mathrm{Hom}(\tilde{\mathfrak{X}}, \mathfrak{Y})$. We choose a point on this latter set, i.e. we choose a map from $\tilde{\mathfrak{X}}$ to $\mathfrak{Y}$, and denote by H the subgroup of $\mathrm{Aut}(\tilde{\mathfrak{X}})$ stabilising this point. This identifies $\mathfrak{Y}$ with the quotient $H\backslash\tilde{\mathfrak{X}}$. An element of $\mathrm{Aut}(\tilde{\mathfrak{X}})$ induces an automorphism of $\mathfrak{Y}$ if and only if it normalises H. Since the elements acting trivially on $\mathfrak{Y}$ are precisely those of H we get an isomorphism between the group of cover automorphisms of $\mathfrak{Y}$ over $\mathfrak{X}$ and NH/H where NH denotes the normaliser of H in $\mathrm{Aut}(\tilde{\mathfrak{X}})$.

The same can be done for covers with a G-action. So we assume a finite group G of automorphisms acts from the left on $\mathfrak{X}$. The following constructions were already sketched in [SGA1, Remarque IX.5.8] and we will fill out some of the details:

For a connected $\mathcal{C}$-cover $p : \mathfrak{Y} \to \mathfrak{X}$ we define the following $\mathcal{C}_G$-cover:

$$\begin{array}{ccc} \mathfrak{Y} \times G & \to & \mathfrak{X} \\ (y, h) & \mapsto & h \cdot p(y) \end{array}$$

and a left G-action on $\mathfrak{Y} \times G$ via

$$\begin{array}{ccccc} G & \times & (\mathfrak{Y} \times G) & \to & \mathfrak{Y} \times G \\ g & , & (y, h) & \mapsto & (y, gh) \end{array}$$

This clearly is a connected object of $\mathcal{C}_G$. The object so associated to $\mathfrak{Y}$ is the same as the fibre product of $\mathfrak{Y}$ with $(\mathfrak{X} \times G)$ with G-action as described above over $\mathfrak{X}$.

Every connected G-cover of $\mathfrak{X} \times G$ is dominated by a G-cover of the form $\mathfrak{Y} \times_{\mathfrak{X}} (\mathfrak{X} \times G)$ where $\mathfrak{Y} \to \mathfrak{X}$ is a connected topological cover. Indeed, forgetting the G-action, a connected G-cover of $\mathfrak{X} \times G$ becomes a cover of $\mathfrak{X}$ consisting of exactly $|G|$ components. If we choose $\mathfrak{Y}$ to dominate each of these components it is not complicated to obtain a G-morphism from $\mathfrak{Y} \times G$ onto the G-cover of $\mathfrak{X} \times G$ we started with.

Given an arbitrary connected G-cover $\mathfrak{Z} \to \mathfrak{X}$ we can form its fibre product with $\mathfrak{X} \times G$. Then we can find a connected component $\mathfrak{Z}'$ (with respect to $\mathcal{C}_G$) of this fibre product that dominates $\mathfrak{Z}$. Since $\mathfrak{Z}'$ is a G-cover of $\mathfrak{X} \times G$ it is dominated by $\tilde{\mathfrak{X}} \times G$. From this we conclude that $\tilde{\mathfrak{X}} \times G$ is a universal cover of $\mathcal{C}_G$.

We define $\tilde{p}_G : \tilde{\mathfrak{X}} \times G \to \mathfrak{X}$ with the G-action as above and denote its group of G-automorphisms by $\mathrm{Aut}(\tilde{\mathfrak{X}} \times G)$. For every point x_0 of $\mathfrak{X}$, this latter group acts from the left on the fibre ${\tilde{p}_G}^{-1}(x_0)$.

Definition 4.4 *For a point x_0 of $\mathfrak{X}$ and a group G that acts by automorphisms on $\mathfrak{X}$ we denote the opposite group of* $\mathrm{Aut}(\mathfrak{X} \times G)$ *by* $\boldsymbol{\pi_1^{\mathrm{top}}(\mathfrak{X}, \mathrm{G}, \mathrm{x_0})}$ *and call it the* **G-fundamental group** *of $\mathfrak{X}$.*

As in the case of the classical fundamental group to give a G-cover is the same as to give a discrete set with a right action of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ on it.

Given a subgroup H of G the fundamental groups classifying covers with actions of H and G are related as follows: We fix a system R of representatives of G/H. We will assume that the class of H is represented by the unit element of G. For a connected $\mathcal{C}_H$-cover $p : \mathfrak{Y} \to \mathfrak{X}$ we define the following $\mathcal{C}_G$-cover:

$$\begin{array}{ccc} \mathfrak{Y} \times R & \to & \mathfrak{X} \\ (y, r) & \mapsto & r \cdot p(y) \end{array}$$

and a left G-action on $\mathfrak{Y} \times R$ via

$$\begin{array}{ccccc} G & \times & (\mathfrak{Y} \times R) & \to & \mathfrak{Y} \times R \\ g & , & (y, r) & \mapsto & (h_g y, r_g h) \end{array}$$

where $g = r_g h_g$ is the unique decomposition of an element of G into a product of an element of H and an element of R. This clearly is a connected object of $\mathcal{C}_G$. The object so associated to $\mathfrak{Y}$ is the same as the fibre product of $\mathfrak{Y}$ with $q : \mathfrak{X} \times R \to \mathfrak{X}$ with the G-action described above. This is an exact functor from $\mathcal{C}_H$ to $\mathcal{C}_G$ and hence defines an injective homomorphism of fundamental groups

$$\pi_1^{\mathrm{top}}(\mathfrak{X},\, H,\, x_0) \;\hookrightarrow\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0).$$

With respect to the action of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ on the fibre $q^{-1}(x_0)$ the image of this homomorphism is the stabiliser of the point $(x_0, 1)$ of $\mathfrak{X} \times R$.

Given an element γ of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ it acts on the fibre $q^{-1}(x_0)$ of the G-cover $q : \mathfrak{X} \times R \to \mathfrak{X}$ by sending $(h^{-1}x_0, h)$ to $(h^{-1}r(\gamma)^{-1}x_0, r(\gamma)h)$. Moreover, if H is a normal subgroup of G then the map that sends γ to $r(\gamma)$ defines a homomorphism from $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ to G/H. This homomorphism is surjective since we can lift the map $x \mapsto g \cdot x$ to the universal cover as explained in [Di, Satz I.8.9]. Hence there exists a short exact sequence

$$1 \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, H,\, x_0) \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \;\to\; G/H \;\to\; 1.$$

In particular, for $H = 1$ we obtain the short exact sequence

$$1 \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, x_0) \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \;\to\; G \;\to\; 1. \qquad (*)$$

To obtain an isomorphism of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ with $\mathrm{Aut}(\tilde{\mathfrak{X}} \times G)$ we have to choose a point on the fibre ${\tilde{p}_G}^{-1}(x_0)$.

If we choose another base point, say x_1 on $\mathfrak{X}$ then the G-fundamental groups with respect to two x_i's are isomorphic. However, such an isomorphism depends on the choice of points $\tilde{x}_i$ in the respective fibres ${\tilde{p}_G}^{-1}(x_i)$, $i = 0, 1$. We will assume that the two $\tilde{x}_i$'s lie on the same topological component of the universal G-cover $\tilde{\mathfrak{X}} \times G$. This means that we choose a path connecting x_0 to x_1. Then the isomorphism

$$\pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \;\cong\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_1)$$

is well-defined up to conjugation by an element of $\pi_1^{\mathrm{top}}(\mathfrak{X}, x_0)$ and the two homomorphisms onto G coming from the short exact sequence $(*)$ are compatible under this isomorphism.

For a G-cover $p : \mathfrak{Y} \to \mathfrak{X}$ and a closed subset of $\mathfrak{A}$ of $\mathfrak{Y}$ we call the subgroup of G fixing $\mathfrak{A}$ pointwise the *inertia group* of $\mathfrak{A}$ (in G):

$$\mathrm{I}_{\mathfrak{A}} \;:=\; \{g \in G \,|\, ga = a,\ \forall a \in \mathfrak{A}\}.$$

The possibly larger subgroup of G fixing $\mathfrak{A}$ but not necessarily pointwise is called the *decomposition group* of $\mathfrak{A}$ (in G):

$$\mathrm{D}_{\mathfrak{A}} \;:=\; \{g \in G \,|\, g(\mathfrak{A}) = \mathfrak{A}\}.$$

The inertia group is always a normal subgroup of the decomposition group.

We choose a point x_1 on $\mathfrak{X}$ and let $\tilde{p}_G : \tilde{\mathfrak{X}} \times G \to \mathfrak{X}$ be the universal G-cover. Then the inertia group I_{x_1} acts on the fibre ${\tilde{p}_G}^{-1}(x_1)$. We choose a point $\tilde{x}_1$ on this fibre. Then we compare the left action of I_{x_1} with the right action of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_1)$ in this point $\tilde{x}_1$. This associates to each element of I_{x_1} an element of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_1)$. Given another point $\tilde{x}_1'$ above x_1 there is a G-automorphism φ that sends $\tilde{x}_1$ to $\tilde{x}_1'$. We assume that $g \cdot \tilde{x}_1 = \tilde{x}_1 \cdot \gamma_g$ for an element g of I_{x_1}. Since φ is G-equivariant we compute

$$g \cdot \tilde{x}_1' \;=\; g \cdot (\varphi(\tilde{x}_1)) \;=\; \varphi(g \cdot \tilde{x}_1) \;=\; \varphi(\tilde{x}_1 \cdot \gamma_g) \;=\; \varphi(\tilde{x}_1) \cdot \gamma_g \;=\; \tilde{x}_1' \cdot \gamma_g.$$

Hence γ_g does not depend on the choice of a point in the fibre above x_1 and it acts like multiplication by g on all points on this fibre. This means that there is a natural injective homomorphism

$$\mathrm{I}_{x_1} \;\hookrightarrow\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_1).$$

If we fix an isomorphism between the G-fundamental groups of $\mathfrak{X}$ with respect to x_0 and x_1 there is the following composition

$$\mathrm{I}_{x_1} \quad\hookrightarrow\quad \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_1) \quad\cong\quad \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \quad\twoheadrightarrow\quad G.$$

Even though the isomorphism in the middle is only well-defined up to conjugation by an element of $\pi_1^{\mathrm{top}}(\mathfrak{X}, x_0)$ the whole composition always coincides with the inclusion map of I_{x_1} into G.

In terms of automorphisms of the universal G-cover $\tilde{p}_G : \tilde{\mathfrak{X}} \times G \to \mathfrak{X}$ we fix a point $\tilde{x}_1$ on the fibre ${\tilde{p}_G}^{-1}(x_1)$. Given an element g of I_{x_1} there is a unique automorphism φ_g of $\tilde{\mathfrak{X}} \times G$ that sends $\tilde{x}_1$ to $g \cdot \tilde{x}_1$. However, this automorphism really depends on the choice of $\tilde{x}_1$.

We now let $\mathfrak{R}$ be a path connected subset of $\mathfrak{X}$ that contains the point x_1. If we forget the G-action for a moment then ${\tilde{p}_G}^{-1}(\mathfrak{R})$ is a disconnected topological cover of $\mathfrak{R}$ if G is non-trivial. We let $\tilde{\mathfrak{R}}$ be a component of ${\tilde{p}_G}^{-1}(\mathfrak{R})$ on $\tilde{\mathfrak{X}} \times \{1\}$. The group $\mathrm{I}_{\mathfrak{R}}$ acts on $\tilde{\mathfrak{R}}$ simply by interchanging the $|\mathrm{I}_{\mathfrak{R}}|$ different but homeomorphic components. We choose a point of $\tilde{\mathfrak{R}}$ above x_1 to obtain an isomorphism of $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ with $\mathrm{Aut}(\tilde{\mathfrak{X}} \times G)$. In this special situation we see that an automorphism corresponding to an element of $\mathrm{I}_{\mathfrak{R}}$ depends only on $\tilde{\mathfrak{R}}$ and not on the particular point lying above x_1. Hence it makes sense to talk about an automorphism of the universal G-cover that is the *inertia automorphism* of a component of ${\tilde{p}_G}^{-1}(\mathfrak{R})$.

We finally want to stress that in general there is no natural way of relating elements of G to cover automorphisms of $\tilde{\mathfrak{X}} \times G$ or elements of the G-fundamental group of $\mathfrak{X}$ since the G-action usually does not respect the fibres. It is only inertia that makes this possible.

Given a G-cover $p : \mathfrak{Y} \to \mathfrak{X}$ there is always an injection of inertia groups $\mathrm{I}_y \subseteq \mathrm{I}_{p(y)}$ for all points $y \in \mathfrak{Y}$. Given a cover $\mathfrak{Z} \to G\backslash\mathfrak{X}$ we can form the fibre product with $\mathfrak{X}$ and obtain a G-cover $p' : \mathfrak{Z} \times_{G\backslash\mathfrak{X}} \mathfrak{X} \to \mathfrak{X}$. All points z on this fibre product fulfill $\mathrm{I}_z = \mathrm{I}_{p'(z)}$. Conversely, if $p : \mathfrak{Y} \to \mathfrak{X}$ is a G-cover that fulfills $\mathrm{I}_y = \mathrm{I}_{p(y)}$ for all points y of $\mathfrak{Y}$ then the quotient by G defines a cover $G\backslash p : G\backslash\mathfrak{Y} \to G\backslash\mathfrak{X}$. Hence there is a one-to-one correspondence

$$\left\{ \text{ covers of } G\backslash\mathfrak{X} \text{ } \right\} \quad\leftrightarrow\quad \left\{ \begin{array}{l} G\text{-covers } p : \mathfrak{Y} \to \mathfrak{X} \text{ such that} \\ \mathrm{I}_y = \mathrm{I}_{p(y)} \text{ for all } y \in \mathfrak{Y} \end{array} \right\}$$

Since this remains true if we assume connectivity on both sides of this correspondence the induced homomorphism of fundamental groups

$$\pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \quad\twoheadrightarrow\quad \pi_1^{\mathrm{top}}(G\backslash\mathfrak{X},\, \bar{x}_0)$$

is surjective. Here, $\bar{x}_0$ denotes the image of x_0 under the quotient map $\mathfrak{X} \to G\backslash\mathfrak{X}$.

For an element g of the inertia group I_{x_1} we denote by $\imath_g$ the image of g in $\pi_1^{\mathrm{top}}(\mathfrak{X},\, G, x_1)$ as constructed above. We let $p : \mathfrak{Y} \to \mathfrak{X}$ be a G-cover with $\mathrm{I}_y = \mathrm{I}_{p(y)}$ for all points of $\mathfrak{Y}$. If we fix a point $\tilde{x}$ on $\tilde{\mathfrak{X}}$ then there is a unique cover automorphism φ_g of $\tilde{\mathfrak{X}} \times G$ such that $\varphi_g \cdot \tilde{x} = \tilde{x} \cdot \imath_g$. By our assumptions on the inertia groups of $\mathfrak{Y}$ this automorphism φ_g will act trivially on $\mathfrak{Y}$. So the subgroup N normally generated by all inertia elements lies in the kernel of the homomorphism from $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)$ onto $\pi_1^{\mathrm{top}}(G\backslash\mathfrak{X}, \bar{x}_0)$. Conversely, the quotient of the universal G-cover by N is a G-cover $q : \mathfrak{Z} \to \mathfrak{X}$ with $\mathrm{I}_z = \mathrm{I}_{q(z)}$ for all points z of $\mathfrak{Z}$. Hence $\pi_1^{\mathrm{top}}(\mathfrak{X}, G, x_0)/N$ is a quotient of $\pi_1^{\mathrm{top}}(G\backslash\mathfrak{X}, \bar{x}_0)$. But this means that N is precisely the kernel we are looking for. So we obtain a short exact sequence

$$1 \;\to\; N \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \;\to\; \pi_1^{\mathrm{top}}(G\backslash\mathfrak{X},\, \bar{x}_0) \;\to\; 1.$$

As a special case we obtain the following: If G acts without fixed points on $\mathfrak{X}$ then $\mathfrak{X} \to G\backslash\mathfrak{X}$ is a regular cover with group G, there are no non-trivial inertia groups and we just get the well-known short exact sequence

$$1 \;\to\; \pi_1^{\mathrm{top}}(\mathfrak{X}, x_0) \;\to\; \pi_1^{\mathrm{top}}(G\backslash\mathfrak{X},\, \bar{x}_0) \;\to\; G \;\to\; 1.$$

4.4 Loops and the orbifold fundamental group

The material of this section should be well-known. However, the author could not find a reference for it.

As in the previous section we let $\mathfrak{X}$ be normal irreducible complex analytic space and G be a finite group of automorphisms of $\mathfrak{X}$. We keep all notations introduced so far.

We will always assume that the quotient space $G\backslash\mathfrak{X}$ is smooth, i.e. a complex manifold. By purity of the branch locus the branch locus $\mathfrak{D}$ of $q : \mathfrak{X} \to G\backslash\mathfrak{X}$ is a divisor, cf. [GR1, Satz 4]. We denote by $\mathfrak{D}_i$, $i = 1, ..., r$ the irreducible components of this divisor, cf. [GR2, Chapter 9.2.2]. We denote by e_i the ramification index of $\mathfrak{D}_i$.

The inertia groups of the components of $q^{-1}(\mathfrak{D}_i)$ for fixed i are conjugate subgroups of G. These components are divisors and so their inertia groups must be cyclic. More precisely, every inertia group of a component of $q^{-1}(\mathfrak{D}_i)$ is abstractly isomorphic to the cyclic group $\mathbb{Z}_{e_i}$.

Given a G-cover $p : \mathfrak{Y} \to \mathfrak{X}$ we form the quotient $G\backslash p : G\backslash\mathfrak{Y} \to G\backslash\mathfrak{X}$. Outside $\bigcup_i \mathfrak{D}_i$ this is a topological cover. This defines a homomorphism from the fundamental group of $G\backslash\mathfrak{X} - \mathfrak{D}$ to the G-fundamental group of $\mathfrak{X}$. If $\mathfrak{Y}$ is connected as a G-cover then its quotient is also connected. We assumed $\mathfrak{X}$ to be normal so also $\mathfrak{Y}$ must be normal and so the same is true for the quotient

$G\backslash\mathfrak{Y}$. After removing the ramification locus of $G\backslash\mathfrak{Y} \to G\backslash\mathfrak{X}$ which has real codimension two this space remains connected since we assumed that our spaces are normal, cf. [GR2, Chapter 7.4.2]. This connectivity result implies that the homomorphism

$$\pi_1^{\mathrm{top}}(G\backslash\mathfrak{X} - \mathfrak{D},\, q(x_0)) \;\twoheadrightarrow\; \pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \qquad (*)$$

is surjective. It remains to compute its kernel.

However, first we want to define a surjective homomorphism

$$\psi \;:\; \pi_1^{\mathrm{top}}(G\backslash\mathfrak{X} - \mathfrak{D},\, q(x_0)) \;\twoheadrightarrow\; G.$$

For this we lift a loop γ in the group on the left to a path in $\mathfrak{X}$ starting at x_0. This lift ends at a point $g \cdot x_0$ where the element $g \in G$ is unique. This defines the homomorphism we are looking for. Of course if we take the pull-back via the morphism $\mathfrak{X} \to G\backslash\mathfrak{X}$ we are in the situation of Section 4.3 where we defined a homomorphism $\pi_1^{\mathrm{top}}(\mathfrak{X},\, G,\, x_0) \twoheadrightarrow G$ in a similar way via lifting elements of the group on the left to the point $x_0 \times 1$ of $\mathfrak{X} \times G$. Chasing through the diagrams we see that the homomorphisms onto G are compatible with the homomorphism $(*)$.

For the divisor $\mathfrak{D}_i$ we define the following loop Γ_i in $\mathfrak{Z} := G\backslash\mathfrak{X} - \mathfrak{D}$: We choose a point w_i on $\mathfrak{D}_i$ that is a smooth point of $\mathfrak{D}$. We let γ_i be a path connecting $q(x_0)$ to w_i inside $\mathfrak{Z}$. We shorten γ_i a little bit before reaching w_i. Then we put a little circle around w_i starting at the end of γ_i. This defines a loop Γ_i based at $q(x_0)$. Such a loop is usually called a *simple loop*.

If we lift this loop to a path based at $x_0 \in \mathfrak{X}$ it "winds around" a component $\mathfrak{R}_i$ of $q^{-1}(\mathfrak{D}_i)$: We choose a small neighbourhood $U(w_i)$ of the point $w_i \in \mathfrak{D}_i$ that we have chosen above. We let $V(w_i)$ be the connected component of $q^{-1}(U(w_i))$ such that the lift of Γ_i to x_0 meets $V(w_i)$. The map $q : \mathfrak{X} \to G\backslash\mathfrak{X}$ looks in local coordinates like

$$\begin{array}{ccc} V(w_i) & \to & U(w_i) \\ (z_1, z_2, \ldots) & \mapsto & ({z_1}^{e_i}, z_2, \ldots) \end{array}$$

where e_i is the ramification index of $\mathfrak{D}_i$. The reason for this is that locally around w_i the map q is a branched Galois cover with group $\mathbb{Z}_{e_i}$ and branch locus $\mathfrak{D}_i$.

In these coordinates $\mathfrak{R}_i$ is given by the equation $z_1 = 0$. The automorphism of $\mathfrak{X}$ induced by the lift of Γ_i to x_0 clearly is the map $x \mapsto \psi(\Gamma_i) \cdot x$. It is clear from this local description that $\mathfrak{R}_i$ must be fixed by $\psi(\Gamma_i)$.

We let $\tilde{p} : \tilde{\mathfrak{X}} \to \mathfrak{X}$ be a universal cover of $\mathfrak{X}$. We choose a point $\tilde{x}_0$ lying above $x_0 \in \mathfrak{X}$. Lifting Γ_i to $\tilde{x}_0$ we get a path that "winds" around a component $\tilde{\mathfrak{R}}_i'$ of $\tilde{p}^{-1}(\mathfrak{R}_i)$. It corresponds to an automorphism of $\tilde{\mathfrak{X}}$ that fixes $\tilde{\mathfrak{R}}_i'$. Via base change to $\mathfrak{X} \times G \to \mathfrak{X}$ we get exactly an element that corresponds to the inertia automorphism of $\tilde{\mathfrak{R}}_i'$ corresponding to $\psi(\Gamma_i)$ as described in Section 4.3.

This automorphism has order e_i and so the image of ${\Gamma_i}^{e_i}$ under $(*)$ must be trivial. In particular, the subgroup normally generated by the ${\Gamma_i}^{e_i}$'s lies in the kernel of $(*)$. We stop for a moment to define a new object:

With respect to the Γ_i and e_i we define the *orbifold fundamental group* with respect to $G\backslash\mathfrak{X}$, $\mathfrak{D}_i$, e_i to be the quotient

$$\pi_1^{\rm orb}\left(G\backslash\mathfrak{X},\,\{\mathfrak{D}_i,\,e_i\},\,q(x_0)\right) \;:=\; \pi_1^{\rm top}\left(G\backslash\mathfrak{X}-\mathfrak{D},\,q(x_0)\right)/\ll {\Gamma_i}^{e_i} \gg .$$

If we choose different set of loops Γ_i' around the $\mathfrak{D}_i$'s as described above then they are conjugate to the original Γ_i's and so this set generates the same normal subgroup. Hence this quotient is well-defined.

This orbifold fundamental group is the opposite automorphism group of some topological cover $\tilde{c}' : \tilde{\mathfrak{Y}} \to G\backslash\mathfrak{X} - \mathfrak{D}$. By what we have said above the homomorphism $(*)$ factors through the orbifold fundamental group and so $\tilde{\mathfrak{Y}}$ dominates $\tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{D})$. For a smooth point w_i on $\mathfrak{D}_i$ we let $U(w_i)$ be an admissible neighbourhood, i.e. a neighbourhood such that $\tilde{c}'^{-1}(U(w_i))$ is a disjoint union of spaces that are homeomorphic to $U(w_i)$. We assume that $\mathfrak{D}_i$ is smooth in $U(w_i)$ so that $U(w_i)-\mathfrak{D}_i$ is homeomorphic to $(\mathbb{C}-\{0\})\times\mathbb{C}^{\dim\mathfrak{X}-1}$. This means that the fundamental group of $U(w_i)-\mathfrak{D}_i$ is isomorphic to $\mathbb{Z}$. It is generated by a loop Γ_i as described above. Looking at this locally we can extend $\tilde{\mathfrak{Y}} \to \tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{D})$ to some map $\bar{\tilde{\mathfrak{Y}}} \to \tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{S})$ where $\mathfrak{S}$ is the finite set of singularities of $\mathfrak{D}$. Since both spaces are locally homeomorphic this is a topological cover map. The space $\tilde{\mathfrak{X}}$ is normal and simply connected. Since $(q\circ\tilde{p})^{-1}(\mathfrak{S})$ is a discrete set of real codimension 4 also $\tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{S})$ is simply connected. Since $\bar{\tilde{\mathfrak{Y}}}$ is a connected topological cover of $\tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{S})$ they must be homeomorphic. Then there is only one way to complete this to a cover of $\tilde{\mathfrak{X}}$: namely to take the trivial cover of $\tilde{\mathfrak{X}}$. So we conclude that $\tilde{\mathfrak{Y}}$ is homeomorphic to $\tilde{\mathfrak{X}} - (q\circ\tilde{p})^{-1}(\mathfrak{D})$ and this means that the homomorphism $(*)$ induces an isomorphism

$$\pi_1^{\rm orb}(G\backslash\mathfrak{X},\,\{\mathfrak{D}_i,\,e_i\},\,q(x_0)) \;\cong\; \pi_1^{\rm top}(\mathfrak{X},\,G,\,x_0).$$

We already noted above that both groups possess surjective homomorphisms onto G that are compatible under this isomorphism.

4.5 The quotient in the topological setup

We let X be smooth projective surface over the complex numbers and $f: X \to \mathbb{P}^2$ be a good generic projection of degree n. From Proposition 2.7 we know that $\mathfrak{S}_n$ acts on $X_{\rm gal}$. With respect to this action and the action of the subgroup $\mathfrak{S}_{n-1}^{(1)}$ we obtain the following two short exact sequences

$$\begin{array}{ccccccccc}
1 & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}) & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\,\mathfrak{S}_n) & \to & \mathfrak{S}_n & \to & 1 \\
 & & || & & \uparrow & & \uparrow & & \\
1 & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}) & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\,\mathfrak{S}_{n-1}^{(1)}) & \to & \mathfrak{S}_{n-1}^{(1)} & \to & 1
\end{array}$$

The arrows upwards are injective. We now fix a universal cover $\widetilde{X_{\mathrm{gal}}}$ of X_{gal} and also do not mention base points unless it is important for our considerations.

The quotient $\mathfrak{S}_n \backslash X_{\mathrm{gal}}$ is isomorphic to $\mathbb{P}^2$ and the inertia groups generate the kernel of the homomorphism from $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\,\mathfrak{S}_n)$ onto $\pi_1^{\mathrm{top}}(\mathfrak{S}_n \backslash X_{\mathrm{gal}})$. Since this latter group is trivial it follows that the inertia groups generate $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\mathfrak{S}_n)$.

The quotient $\mathfrak{S}_{n-1}^{(1)} \backslash X_{\mathrm{gal}}$ is isomorphic to X. So the kernel of the surjective homomorphism onto the fundamental groups of X is the the subgroup normally generated by the inertia groups contained in $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\mathfrak{S}_{n-1}^{(1)})$.

By Proposition 2.12 the ramification divisor R_{gal} of $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to \mathbb{P}^2$ is the union of the curves R_τ where τ runs through the transpositions of $\mathfrak{S}_n$. We denote by $\tilde{p} : \widetilde{X_{\mathrm{gal}}} \to X_{\mathrm{gal}}$ the universal cover of X_{gal}. Then we let $\tilde{R}_\tau$ be a connected component of $\tilde{p}^{-1}(R_\tau)$. We have seen in the previous section that there is a unique inertia automorphism of the universal $\mathfrak{S}_n$-cover $\widetilde{X_{\mathrm{gal}}} \times \mathfrak{S}_n$ that sends $\tilde{R}_\tau \times \{1\}$ to $\tilde{R}_\tau \times \{\tau\}$. Since the inertia group of R_τ is $\mathbb{Z}_2$ this automorphism is the only non-trivial inertia automorphism of $\tilde{R}_\tau$.

We let τ_1 and τ_2 be two transpositions of $\mathfrak{S}_n$ and choose two components $\tilde{R}_{\tau_1}$ and $\tilde{R}_{\tau_2}$ of $\tilde{p}^{-1}(R_{\tau_1})$ and $\tilde{p}^{-1}(R_{\tau_2})$, respectively. For the non-trivial inertia elements r_1 and r_2 of their inertia groups we set (cf. Definition 2.11)

$$c(r_1,\,r_2) := \begin{cases} 1 & \text{if } \tau_1 = \tau_2 \\ r_1 r_2 r_1^{-1} r_2{}^{-1} & \text{if } \tau_1 \text{ and } \tau_2 \text{ are disjoint} \\ r_1 r_2 r_1 r_2{}^{-1} r_1{}^{-1} r_2{}^{-1} & \text{if } \tau_1 \text{ and } \tau_2 \text{ are cuspidal.} \end{cases}$$

Then we define C^{proj} to be the subgroup normally generated by all the $c(r_1, r_2)$'s inside $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\mathfrak{S}_n)$ where the τ_i's run through all transpositions of $\mathfrak{S}_n$ and the r_i's run through all inertia groups of all components of the $\tilde{p}^{-1}(R_{\tau_i})$'s.

Lemma 4.5 *The subgroup C^{proj} is contained in $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}})$ and in the following kernels:*

$$\begin{array}{llll}
\ker(& \pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\mathfrak{S}_n) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(\mathbb{P}^2) &) \\
\ker(& \pi_1^{\mathrm{top}}(X_{\mathrm{gal}},\mathfrak{S}_{n-1}^{(1)}) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(X) &)
\end{array}$$

The proof is completely analogous to the proof of Lemma 4.1 and therefore left to the reader. □

Hence the homomorphisms from $\pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n)$ onto $\pi_1^{\text{top}}(\mathbb{P}^2)$ and the map from $\pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_{n-1}^{(1)})$ onto $\pi_1^{\text{top}}(X)$ factor over the quotient by C^{proj}. Moreover, we get the following two short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^{\text{top}}(X_{\text{gal}}) & \to & \pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n) & \to & \mathfrak{S}_n & \to & 1 \\ & & \downarrow & & \downarrow & & || & & \\ 1 & \to & \pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}} & \to & \pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n)/C^{\text{proj}} & \to & \mathfrak{S}_n & \to & 1 \quad (*) \end{array}$$

where the arrows downwards are surjective.

Proposition 4.6 *We can split the short exact sequence* $(*)$ *using inertia groups. With respect to this splitting there are the following isomorphisms*

$$\begin{array}{ccccc} (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}}) & / & (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}})_{\mathfrak{S}_n} & \cong & \pi_1^{\text{top}}(\mathbb{P}^2) \,=\, \{1\} \\ (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}}) & / & (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}})_{\mathfrak{S}_{n-1}^{(1)}} & \cong & \pi_1^{\text{top}}(X) \end{array}$$

where the notations are the ones introduced in Section 3.1

If Question 2.14 has an affirmative answer for the universal cover $\widetilde{X_{\text{gal}}}$ *of* X_{gal} *then the group* C^{proj} *is trivial.*

PROOF. The proof is analogous to the one of Proposition 4.2:

For every transposition $(1\,k)$ we choose a component of $\tilde{p}^{-1}(R_{(1\,k)})$ and denote by r_k the non-trivial element of its inertia group. We denote by $\bar{r}_k$ the image of r_k inside $\pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n)/C^{\text{proj}}$. As in the proof of Proposition 4.2 we conclude that these $\bar{r}_k$'s fulfill the Coxeter relations of the symmetric group and so they provide us with a splitting $s : \mathfrak{S}_n \to \pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n)/C^{\text{proj}}$.

As in the proof of Proposition 4.2 there are the following equalities for the kernel N of the homomorphism from $\pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n)$ onto $\pi_1^{\text{top}}(\mathbb{P}^2)$:

$$\begin{array}{rcl} N & = & (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}})_{\mathfrak{S}_n} \cdot s(\mathfrak{S}_n) \\ \text{and} \quad N \cap (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}})_{\mathfrak{S}_n} & = & (\pi_1^{\text{top}}(X_{\text{gal}})/C^{\text{proj}})_{\mathfrak{S}_n} \end{array}$$

Applying the second isomorphism theorem of group theory we obtain the first statement. Again, we leave the second identity to the reader.

Now suppose that the components of $\tilde{p}^{-1}(R_{\text{gal}})$ fulfill the connectivity properties of Question 2.14 with respect to the universal cover $\tilde{p} : \widetilde{X_{\text{gal}}} \to X_{\text{gal}}$.

For two disjoint transpositions τ_1 and τ_2 we choose components $\tilde{R}_1$ and $\tilde{R}_2$ of $\tilde{p}^{-1}(R_1)$ and $\tilde{p}^{-1}(R_2)$, respectively. We let r_1 and r_2 be the non-trivial elements of their inertia groups. We know that these components intersect in a point z. There is an inclusion of $\mathbb{Z}_2 \times \mathbb{Z}_2$ into $\pi_1^{\text{top}}(X_{\text{gal}}, \mathfrak{S}_n, \tilde{p}(z))$. This group is generated by r_1 and r_2. Hence these two elements commute and $c(r_1, r_2)$ is equal to 1.

If τ_1 and τ_2 have exactly one index in common then there is a point with inertia group $\mathfrak{S}_3$ that is generated by r_1 and r_2. So there is a triple commutator relation between r_1 and r_2 and so also $c(r_1, r_2) = 1$ holds true in this case.

So if Question 2.14 has an affirmative answer for the universal cover $\widetilde{X_{\mathrm{gal}}}$ of X_{gal} then all the $c(r_1, r_2)$'s are equal to 1 and so C^{proj} is trivial. $\square$

Again, everything said so far can also be done in the affine setup. We then define C^{aff} to be the normal subgroup of $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}, \mathfrak{S}_n)$ defined by the $c(r_1, r_2)$'s where the r_i's run through the inertia groups corresponding to some universal $\mathfrak{S}_n$-cover of $X_{\mathrm{gal}}^{\mathrm{aff}}$. We then get

Theorem 4.7 *Let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n with Galois closure X_{gal}. Then there are surjective homomorphisms*

$$\begin{array}{ccccc} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \twoheadrightarrow & \mathcal{K}(\pi_1^{\mathrm{top}}(X),\, n) \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \twoheadrightarrow & \mathcal{K}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}),\, n). \end{array}$$

If Question 2.14 has an affirmative answer for the universal cover of $X_{\mathrm{gal}}^{\mathrm{aff}}$ then both C^{aff} and C^{proj} are trivial.

If Question 2.14 has an affirmative answer for the universal cover of X_{gal} then at least C^{proj} is trivial.

Again, even if C^{aff} is trivial we cannot expect these surjective homomorphisms to be isomorphisms. We refer to Theorem 6.2 for details.

Corollary 4.8 *For a good generic projection $f : X \to \mathbb{P}^2$ of degree n there are surjective and non-canonical homomorphisms*

$$\begin{array}{ccc} H_1(X_{\mathrm{gal}}, \mathbb{Z}) & \twoheadrightarrow & (H_1(X, \mathbb{Z}))^{n-1} \\ H_1(X_{\mathrm{gal}}^{\mathrm{aff}}, \mathbb{Z}) & \twoheadrightarrow & \left(H_1(X^{\mathrm{aff}}, \mathbb{Z})\right)^{n-1} \end{array} .$$

PROOF. From Morse theory ([Mil]) it is known that smooth affine and smooth projective varieties are CW-complexes. So we can apply Hurewicz's theorem that $H_1(-,\, \mathbb{Z})$ is isomorphic to the abelianised fundamental group.

Thus our statement follows from the fact that $\mathcal{K}(-, n)$ for $n \geq 3$ commutes with abelianisation by Proposition 3.8 and the computation of $\mathcal{K}(-, n)$ for abelian groups given by Corollary 3.5. $\square$

5 A generalised symmetric group

Seht doch hinab! Im Mondschein auf den Gräbern
Hockt eine wild-gespenstische Gestalt!
Ein Aff ist's! Hört ihr, wie sein Heulen
Hinausgellt in den süßen Duft des Lebens?

5.1 Definition of $\mathcal{S}_n(\mathrm{d})$

We let τ_k be the transposition $(k\,k+1)$ of $\mathfrak{S}_n$. From the theory of Coxeter groups (cf. also Section 5.6) it is known that $\mathfrak{S}_n$ admits a presentation as

$$\mathfrak{S}_n \quad = \quad \left\langle \tau_k,\, k=1,...,n-1 \,|\, {\tau_k}^2,\, (\tau_k\tau_{k+1})^3,\, (\tau_k\tau_j)^2\, \forall |k-j|\geq 2 \right\rangle .$$

Let $d\geq 1$ and $n\geq 3$ be natural numbers. We want to construct a generalised symmetric group where we have d copies of the transposition $(1\,2)$. For this we let $s_1,...,s_d$ be free generators of the free group $\mathfrak{F}_d$ of rank d. Then we define the group

$$\tilde{\mathcal{S}}_n(d) \quad := \quad \left(\mathfrak{F}_d \,*\, \mathfrak{S}^{(1)}_{n-1}\right)/R$$

where R is the subgroup normally generated by the following elements

$$\begin{array}{rl} {s_i}^2 & \text{for } i=1,...,d \\ (s_i\cdot\tau_2)^3 & \text{for } i=1,...,d \\ (s_i\cdot\tau_k)^2 & \text{for } k\geq 3 \text{ and } i=1,...,d. \end{array}$$

The reader will identify this group as the d-fold amalgamated sum of $\mathfrak{S}_n$ with itself where we amalgamate the subgroup $\mathfrak{S}^{(1)}_{n-1}$ in every summand.

Every summand has a map (the identity) onto $\mathfrak{S}_n$ that is compatible with the subgroup that is amalgamated. These homomorphism patch together to a homomorphism ψ onto $\mathfrak{S}_n$. Sending $\mathfrak{S}_n$ via the identity to the first summand we obtain a splitting φ of ψ.

But we still want more relations to hold true: We define

$$\mathcal{S}_n(d) \quad := \quad \tilde{\mathcal{S}}_n(d)/R'$$

where R' is the subgroup normally generated by the following elements:

$$\begin{array}{ll} (\varphi(\sigma)s_i\varphi(\sigma)^{-1}\cdot s_j)^2 & \text{if } \sigma(1\,2)\sigma^{-1} \text{ and } (1\,2) \text{ are nodal transpositions} \\ (\varphi(\sigma)s_i\varphi(\sigma)^{-1}\cdot s_j)^3 & \text{if } \sigma(1\,2)\sigma^{-1} \text{ and } (1\,2) \text{ are cuspidal transpositions} \end{array}$$

The homomorphisms ψ and φ induce homomorphisms on the quotient $\mathcal{S}_n(d)$ that we will call by abuse of notation again by ψ and φ.

5.2 The connection with $\mathcal{E}(- ,\mathrm{n})$

Before dealing with the general situation we do the cases $d \leq 2$ first:

For $d = 1$ we clearly have $\mathcal{S}_n(1) \cong \mathfrak{S}_n$ for all $n \geq 2$.

Proposition 5.1 *For $n \geq 2$ there is an isomorphism*

$$\begin{array}{ccc} \mathcal{S}_n(2) & \cong & \mathcal{E}(\mathbb{Z}, n) \\ s_1 & \mapsto & (1\,2) \\ s_2 & \mapsto & (1, -1, 0, ..., 0) \cdot (1\,2) \end{array}$$

compatible with the respective split surjections onto $\mathfrak{S}_n$.

PROOF. We consider the following elements of $\mathcal{S}_n(2)$

$$\begin{array}{cclc} a & := & (2\,n)(s_2(1\,2))(2\,n)^{-1} \cdot (1\,n) & \\ \tau_k & := & (k\,k+1) & k = 1, ..., n-1. \end{array}$$

The affine reflection group $\tilde{\mathsf{A}}_{n-1}$ has the following presentation, c.f. Section 5.6

$$\begin{array}{cccl} W(\tilde{\mathsf{A}}_{n-1}) & := & \langle \alpha, \tau_k | & {\tau_k}^2,\, (\tau_k\tau_{k+1})^2,\, (\tau_k\tau_j)^2 \text{ for } |k-j| \geq 2, \\ & & & \alpha^2,\, (\alpha\tau_1)^3,\, (\alpha\tau_{n-1})^3,\, (\alpha\tau_k)^2 \text{ for } k \neq 1, n-1 \rangle. \end{array}$$

We define a map $\tilde{\varphi} : W(\tilde{\mathsf{A}}_{n-1}) \to \mathcal{S}_n(2)$ by sending α to a and τ_k to τ_k for all k. The relations inside $W(\tilde{\mathsf{A}}_{n-1})$ also hold true for the corresponding elements in the image i.e. $\tilde{\varphi}$ extends to a homomorphism. In a similar fashion we define a homomorphism in the opposite direction being the inverse of $\tilde{\varphi}$. Hence $\tilde{\varphi}$ is an isomorphism.

Finally, we identify $W(\tilde{\mathsf{A}}_{n-1})$ with $\mathcal{E}(\mathbb{Z}, n)$ using the description given in Corollary 3.6 or Example 5.26. □

Remark 5.2 *There is a general "Coxeter flavour" in connection with $\mathcal{E}(-, n)$. We refer to Section 5.6 for some examples and details.*

We let $\mathfrak{F}_{d-1}$ be the free group of rank $d-1$ freely generated by elements $f_2, ..., f_d$. We denote by θ the action of $\mathfrak{S}_n$ on ${\mathfrak{F}_{d-1}}^n$ given by permuting the factors. We recall that we constructed $\mathcal{E}(-, n)$ using such a θ in Section 3.1.

We want to define a map

$$\begin{array}{ccccl} \phi & : & \mathcal{S}_n(d) & \to & {\mathfrak{F}_{d-1}}^n \rtimes_\theta \mathfrak{S}_n \\ & & s_1 & \mapsto & (1\,2) \\ & & s_a & \mapsto & (f_a, {f_a}^{-1}, 1, ..., 1) \cdot (1\,2) \quad \forall a = 2, ..., d \\ & & \varphi(\sigma) & \mapsto & \sigma \quad \forall \sigma \in \mathfrak{S}_n \end{array}$$

where φ is the splitting that comes together with $\mathcal{S}_n(d)$. Since we have fixed the splitting φ of ψ we consider $\mathfrak{S}_n$ as a subgroup of $\mathcal{S}_n(d)$ and do not mention φ any further. The content of the following theorem is that this map ϕ is not only a homomorphism but also injective with image $\mathcal{E}(\mathfrak{F}_{d-1}, n)$:

Theorem 5.3 *For $n \geq 5$ there exists an isomorphism*

$$\begin{array}{ccccc} \phi & : & \mathcal{S}_n(d) & \cong & \mathcal{E}(\mathfrak{F}_{d-1},\, n) & \leq \mathfrak{F}_{d-1}{}^n \rtimes_\theta \mathfrak{S}_n \\ & & s_1 & \mapsto & (1\,2) & \\ & & s_a & \mapsto & (f_a, f_a{}^{-1}, 1, ..., 1)\,(1\,2) & \forall a \geq 2 \end{array}$$

compatible with the respective split surjections onto $\mathfrak{S}_n$.

PROOF. First we have to check that ϕ extends to a homomorphism. For this we only have to check that all relations of $\mathcal{S}_n(d)$ hold inside the image. These calculations are straight forward and are done in Lemma 5.5.

Also, we see from Lemma 5.5 that the image of ϕ is precisely $\mathcal{E}(\mathfrak{F}_{d-1}, n)$.

Fo $a = 2, ..., d$ and $i, j = 1, ..., n$ we define:

$$f_a{}^{ij} \;:=\; (1, ..., 1, \underbrace{f_a}_{i\text{.th position}}, 1, ..., 1, \underbrace{f_a{}^{-1}}_{j\text{.th position}}, 1, ..., 1) \;\in\; \mathfrak{F}_d{}^n.$$

These elements generate $\mathcal{K}(\mathfrak{F}_{d-1}, n)$ as can be seen from applying Lemma 3.1 using transpositions as generating set for $\mathfrak{S}_n$.

We want to define a homomorphism from $\mathcal{K}(\mathfrak{F}_{d-1}, n)$ to $\mathcal{S}_n(d)$ by sending

$$\begin{array}{ccccc} \hat{\phi} & : & \mathcal{K}(\mathfrak{F}_{d-1}, n) & \to & \mathcal{S}_n(d) \\ & & f_a{}^{ii} & \mapsto & 1 \\ & & f_a{}^{ij} & \mapsto & (1\,i)(2\,j) \cdot (s_a(1\,2)) \cdot (2\,j)^{-1}(1\,j)^{-1} \quad i \neq j \end{array}$$

From Proposition 5.6 we know all the relations that hold between the $f_a{}^{ij}$ inside $\mathcal{K}(\mathfrak{F}_{d-1}, n)$. The relations $(*2)$ and $(*3)$ hold true in $\mathcal{S}_n(d)$ by the relations coming from cuspidal transpositions. The relations $(*4)$ hold true because of the relations coming from nodal transpositions. We leave the details to the reader.

By definition ϕ is the identity when restricted to $\mathfrak{S}_n$. To show that $\hat{\phi}$ extends to a homomorphism from $\mathcal{K}(\mathfrak{F}_{d-1}, n) \rtimes_\theta \mathfrak{S}_n$ to $\mathcal{S}_n(d)$ we only have to show that $\hat{\phi}$ is $\mathfrak{S}_n$-equivariant with respect to the $\mathfrak{S}_n$-action given by conjugation in both groups. We leave it to the reader to show that for $\sigma \in \mathfrak{S}_n$

$$\begin{array}{cl} & \sigma \cdot f_a{}^{ij} \cdot \sigma^{-1} \\ = & f_a{}^{\sigma^{-1}(i)\,\sigma^{-1}(j)} \\ & \sigma \cdot (1\,i)(2\,j) \cdot (s_a(1\,2)) \cdot (2\,j)^{-1}(1\,j)^{-1} \cdot \sigma^{-1} \\ = & (1\,\sigma^{-1}(i))(2\,\sigma^{-1}(j)) \cdot (s_a(1\,2)) \cdot (2\,\sigma^{-1}j)^{-1}(1\,\sigma^{-1}(j))^{-1} \end{array}$$

holds true proving $\mathfrak{S}_n$-equivariance.

Hence, there is a homomorphism from $\mathcal{E}(\mathfrak{F}_{d-1}, n)$ to $\mathcal{S}_n(d)$ prolonging $\hat{\phi}$ and compatible with the split surjections onto $\mathfrak{S}_n$. Since ϕ is surjective and $\hat{\phi} \circ \phi(s_a) = s_a$ for all a it follows that ϕ is an isomorphism. □

Remark 5.4 *The really hard part of this proof is Proposition 5.6. It says that the relations of $\mathcal{K}(\mathfrak{F}_d, n)$ are only some "obvious" ones, i.e. a certain set of commutator relations.*

The author's original proof used a Reidemeister-Schreier rewriting process to obtain a presentation of the subgroup $\mathcal{E}(\mathfrak{F}_{d-1}, n)$ of ${\mathfrak{F}_{d-1}}^n \rtimes_\theta \mathfrak{S}_n$. However, since the subgroup has infinite index in the ambient group he obtained an infinite set of relations. The computations were a ten page flow of quite messy calculations.

Meanwhile, [RTV] appeared and the author decided to copy their proof.

Lemma 5.5 *Let G be an arbitrary group and $\vec{g}_i$, $i = 1, 2$ two elements of $\mathcal{K}(G, n)$. We define*

$$s_i \;:=\; \vec{g}_i\,(1\,2)\,\vec{g}_i^{\,-1},\; i = 1, 2$$

Then the following relations hold inside $\mathcal{E}(G, n)$

$$\begin{array}{rl} {s_i}^2 & i = 1, 2 \\ (s_i \cdot \tau)^2 & \text{if } \tau \text{ and } (1\,2) \text{ are nodal transpositions} \\ (s_i \cdot \tau)^3 & \text{if } \tau \text{ and } (1\,2) \text{ are cuspidal transpositions} \\ (\sigma s_i \sigma^{-1} \cdot s_j)^2 & \text{if } \sigma(1\,2)\sigma^{-1} \text{ and } (1\,2) \text{ are nodal transpositions} \\ (\sigma s_i \sigma^{-1} \cdot s_j)^3 & \text{if } \sigma(1\,2)\sigma^{-1} \text{ and } (1\,2) \text{ are cuspidal transpositions.} \end{array}$$

If $n \geq 3$ and if the elements $g_1, ..., g_s$ generate G then $\mathcal{E}(G, n)$ is generated by $[(g_i, 1, ..., 1), (1\,2)]$ and an arbitrary generating set of $\mathfrak{S}_n$.

PROOF. The first relation is straight forward from Lemma 3.1. Furthermore it allows us to view the remaining relations as commutator relations or triple commutator relations, respectively.

We do the computations inside $G^n \rtimes \mathfrak{S}_n$ as usual. We set $\tau = (3\,4)$ and $\vec{g} = (g_1, g_2, ..., g_n) \in G^n$, and check that $\vec{g}(1\,2)\vec{g}^{-1}$ and τ commute:

$$\begin{array}{rl} & \left((\vec{g}(1\,2)\vec{g}^{-1}) \cdot \tau\right)^2 \\ = & [\vec{g}(1\,2)\vec{g}^{-1},\, \tau] \\ = & \vec{g}(1\,2)\vec{g}^{-1} \cdot \tau \left((g_1 g_2^{-1}, g_2 g_1^{-1}, 1, ..., 1)^{-1} (1\,2)^{-1}\right) \tau^{-1} \\ = & \vec{g}(1\,2)\vec{g}^{-1} \cdot \tau \left((g_1 g_2^{-1}, g_2 g_1^{-1}, 1, ..., 1)^{-1}\right) \tau^{-1} (1\,2)^{-1} \\ = & \vec{g}(1\,2)\vec{g}^{-1} \cdot (g_1 g_2^{-1}, g_2 g_1^{-1}, 1, ..., 1)^{-1} (1\,2)^{-1} \\ = & \vec{g}(1\,2)\vec{g}^{-1} \cdot \left(\vec{g}(1\,2)\vec{g}^{-1}\right)^{-1} \\ = & 1 \end{array}$$

We leave the remaining relations to the reader.

We have already seen in Lemma 3.1 that $\mathcal{E}(G, n)$ is generated by $\mathfrak{S}_n$ and all elements of the form $(g, g^{-1}, 1, ..., 1)$. Let $g_1, ..., g_s$ be a generating set for G. We define $\vec{g}_i := (g_i, 1, ..., 1)$ and compute for $n \geq 3$

$$[\vec{g}_i, (1\,3)] \cdot [\vec{g}_j, (1\,2)] \cdot [\vec{g}_i, (1\,3)] = (g_i g_j,\, (g_i g_j)^{-1}, 1, ..., 1)$$

So we get all elements $(g, g^{-1}, 1, ..., 1)$ from the set $[\vec{g}_i, (1\,2)]$ and $\mathfrak{S}_n$. □

Proposition 5.6 (Rowen, Teicher, Vishne) *We let $\mathfrak{F}_d$ be the free group of rank d and assume that it is freely generated by elements $f_1, ..., f_d$. We set:*

$$f_a{}^{ij} \quad := \quad (1, ..., 1, \underbrace{f_a}_{i.th\ position}, 1, ..., 1, \underbrace{f_a{}^{-1}}_{j.th\ position}, 1, ..., 1) \quad \in \quad \mathfrak{F}_d{}^n$$

If $n \geq 2$ then $\mathcal{K}(\mathfrak{F}_d, n)$ is generated by $f_a{}^{ij}$ with $a = 1, ..., d$ and $i, j = 1, ..., n$. And if $n \geq 5$ then all relations inside $\mathcal{K}(\mathfrak{F}_d, n)$ follow from the following relations:

$$\begin{array}{lcll} f_a{}^{ii} & = & 1 & (*1) \\ f_a{}^{ij} \cdot f_a{}^{jk} & = & f_a{}^{ik} & (*2) \\ f_a{}^{ik} \cdot f_a{}^{ij} & = & f_a{}^{ik} & (*3) \\ \left[f_a{}^{ij},\, f_b{}^{kl}\right] & = & 1 \quad \textit{if } i, j, k, l \textit{ are all different.} & (*4) \end{array}$$

In other words we have a finite presentation of $\mathcal{K}(\mathfrak{F}_d, n)$ for $n \geq 5$.

PROOF. The proof is taken from [RTV, Theorem 5.7]. However, we adapted the notations to our situation.

First of all, the $f_a{}^{ij}$'s generate $\mathcal{K}(\mathfrak{F}_d, n)$. This follows from Lemma 3.1 applied to the generating set f_i of $\mathfrak{F}_d$ and taking as generating set for $\mathfrak{S}_n$ the set of all transpositions.

We leave it to the reader to show that the relations given in the statement of Proposition 5.6 hold true in $\mathfrak{F}_d{}^n$ and hence in $\mathcal{K}(\mathfrak{F}_d, n)$.

We define $K_{d,n}$ to be the group generated by elements $f_a{}^{ij}$ with $a = 1, ..., d$ and $i, j = 1, ..., n$ subject to the relations given by Proposition 5.6. We have shown above that there is a surjective homomorphism from $K_{d,n}$ onto $\mathcal{K}(\mathfrak{F}_d, n)$.

Next, we define $K^*_{d,n}$ to be the group generated by elements

$$f_a{}^{ij} \quad \text{and} \quad t_a \quad \text{with } a = 1, .., d,\ i, j = 1, ..., n$$

subject to the relations of $K_{d,n}$ and the relations

$$\begin{array}{lcll} [t_a,\, f_b{}^{ij}] & = & \left[f_a{}^{nk},\, f_b{}^{ij}\right] \quad k \neq i, j & (\dagger 1) \\ [t_a,\, t_b] & = & [f_a{}^{ni},\, f_b{}^{nj}] \quad i \neq j \text{ and } i, j \neq n & (\dagger 2) \end{array}$$

Then we define the following map

$$\begin{array}{rccc} \mu \ : & K^*_{d,n} & \to & \mathfrak{F}_d{}^n \\ & t_a & \mapsto & f_a{}^n \\ & f_a{}^{ij} & \mapsto & (f_a{}^j)^{-1} f_a{}^i \end{array}$$

where $f_a{}^i$ denotes the element $(1, ..., 1, f_a, 1, ..., 1)$ of $\mathfrak{F}_d{}^n$ having its non-trivial entry in the i.th position. By Lemma 5.7 this map μ defines an isomorphism of groups.

So we obtain the following commutative diagram

$$\begin{array}{ccc} K_{d,n} & \rightarrow & K^*_{n,d} \\ \downarrow & & \downarrow\cong \\ \mathcal{K}(\mathfrak{F}_d, n) & \hookrightarrow & \mathfrak{F}_d{}^n \end{array}$$

We already know that the map from $K_{d,n}$ to $\mathcal{K}(\mathfrak{F}_d, n)$ is surjective. To show that it is also injective it is enough to prove that the homomorphism from $K_{d,n}$ to $K^*_{d,n}$ is injective.

To achieve this we define a series of groups lying in between $K_{d,n}$ and $K^*_{d,n}$: We define $K^{\leq k}_{d,n}$ to be the group generated by $K_{d,n}$ and the elements t_a with $a \leq k$ subject to the relations $(\dagger 1)$ and $(\dagger 2)$. Of course, only those relations that involve t_a's and t_b's with $a, b \leq k$ are imposed. Hence we obtain the following groups and homomorphisms

$$K_{n,d} = K^{\leq 0}_{n,d} \rightarrow K^{\leq 1}_{n,d} \rightarrow \ldots \rightarrow K^{\leq d}_{n,d} = K^*_{n,d}.$$

By Lemma 5.8 each of these homomorphisms is injective and so the composite homomorphism from $K_{n,d}$ to $K^*_{n,d}$ is injective. □

Lemma 5.7 *The map*

$$\mu \ : \ K^*_{n,d} \ \rightarrow \ \mathfrak{F}_d{}^n$$

defined in the proof of Proposition 5.6 is an isomorphism of groups.

PROOF. It is clear that μ defines a surjective homomorphism.

We define a map $\hat{\mu}$ via

$$\begin{array}{cccc} \hat{\mu} & : & \mathfrak{F}_d{}^n & \rightarrow & K^*_{n,d} \\ & & f_a{}^n & \mapsto & t_a \\ & & f_a{}^i & \mapsto & t_a \cdot f_a{}^{in} \end{array}$$

If we can show that $\hat{\mu}$ defines a homomorphism of groups it will be the inverse of μ and it follows that μ is an isomorphism.

The group $\mathfrak{F}_d{}^n$ is generated by the elements $f_a{}^i$ with $a = 1, \ldots, d$ and $i = 1, \ldots, n$ subject to the commutator relations $[f_a{}^i,\, f_b{}^j] = 1$ for all $i \neq j$.

First we establish two further sets of relations that hold true inside $K^*_{d,n}$:

$$\begin{array}{lclll} f_a{}^{in} \cdot t_b \cdot (f_a{}^{in})^{-1} & = & t_a{}^{-1} \cdot t_b \cdot t_a & i \neq n & (\dagger 3) \\ f_a{}^{in} \cdot f_b{}^{jn} \cdot (f_a{}^{in})^{-1} & = & t_a{}^{-1} \cdot f_b{}^{jn} \cdot t_a & i \neq j, \text{ and } i, j \neq n & (\dagger 4) \end{array}$$

The relation $(\dagger 3)$ can be seen by applying $(\dagger 2)$ to the right hand side of $(\dagger 1)$ with $j = n$. The relation $(\dagger 4)$ is only a reformulation of $(\dagger 1)$.

First, assume that $i = n$. Then

$$\begin{aligned}
& \hat{\mu}([f_a{}^n,\ f_b{}^j]) \\
= \;& t_a \cdot t_b f_b{}^{jn} \cdot t_a{}^{-1} \cdot (t_b f_b{}^{jn})^{-1} \\
= \;& t_a \cdot t_b \underbrace{f_b{}^{jn} \cdot t_a{}^{-1} \cdot (f_b{}^{jn})^{-1}}_{\text{apply } (\dagger 3)} t_b{}^{-1} \\
= \;& t_a t_b \cdot t_b{}^{-1} t_a{}^{-1} t_b \cdot t_b{}^{-1} \\
= \;& 1
\end{aligned}$$

Now assume that $i \neq n$. Then

$$\begin{aligned}
& \hat{\mu}([f_a{}^i,\ f_b{}^j]) \\
= \;& t_a f_a{}^{in} \cdot t_b f_b{}^{jn} \cdot (t_a f_a{}^{in})^{-1} \cdot (t_b f_b{}^{jn})^{-1} \\
= \;& t_a \underbrace{f_a{}^{in} t_b \cdot ((f_a{}^{in})^{-1}}_{\text{apply } (\dagger 3)} f_a{}^{in}) \cdot f_b{}^{jn} (f_a{}^{in})^{-1} \underbrace{t_a{}^{-1} (f_b{}^{jn})^{-1} \cdot (t_a}_{\text{apply } (\dagger 4)} t_a{}^{-1}) \cdot t_b{}^{-1} \\
= \;& t_a \cdot t_a{}^{-1} t_b t_a \cdot f_a{}^{in} f_b{}^{jn} (f_a{}^{in})^{-1} \cdot f_a{}^{in} (f_b{}^{jn})^{-1} (f_a{}^{in})^{-1} \cdot t_a{}^{-1} t_b{}^{-1} \\
= \;& 1
\end{aligned}$$

Hence $\hat{\mu}$ defines a homomorphism and so we are done. □

Lemma 5.8 *Keeping the notations introduced in the proof of Proposition 5.6 there is an isomorphism*

$$K_{n,d}^{\leq k} \;\cong\; K_{n,d}^{\leq k-1} \rtimes \mathbb{Z}$$

where the infinite cyclic group $\mathbb{Z}$ is generated by t_k. In particular, the map from $K_{n,d}^{\leq k-1}$ to $K_{n,d}^{\leq k}$ considered in the proof of Proposition 5.6 is injective.

PROOF. We want to define a map from $K_{n,d}^{\leq k-1}$ to itself via

$$\begin{array}{rcccl}
\vartheta \;:\; K_{n,d}^{\leq k-1} & \rightarrow & K_{n,d}^{\leq k-1} & \\
f_a{}^{ij} & \mapsto & f_k{}^{nm} \cdot f_a{}^{ij} \cdot (f_k{}^{nm})^{-1} & m \neq i, j, n \\
t_a & \mapsto & f_k{}^{nm} \cdot t_a \cdot (f_k{}^{nm})^{-1} & m \neq n
\end{array}$$

First we have to show that ϑ does not depend on the choice of m in the definition of ϑ: For the definition of $\vartheta(f_a{}^{ij})$ this means we have to check that for $m, m' \neq i, j, n$

$$f_k{}^{nm} \cdot f_a{}^{ij} \cdot (f_k{}^{nm})^{-1} \;=\; f_k{}^{nm'} \cdot f_a{}^{ij} \cdot (f_k{}^{nm'})^{-1}$$

holds true. If $i, j \neq n$ then both expressions are equal to $f_a{}^{ij}$ by relation $(*4)$. If $i = n$ then we conjugate this expression with $f_k{}^{nm'}$ and after applying $(*2)$ we are done since $f_k{}^{mm'}$ and $f_a{}^{ij}$ commute using $(*4)$.

For the definition of $\vartheta(t_a)$ we have to check that for $m, m' \neq n$

$$ {f_k}^{nm} \cdot t_a \cdot ({f_k}^{nm})^{-1} \;=\; {f_k}^{nm'} \cdot t_a \cdot ({f_k}^{nm'})^{-1} $$

holds true. We conjugate by ${f_k}^{nm'}$ and then we are done since t_a and ${f_k}^{mm'}$ commute by $(\dagger 1)$.

Hence the definition of ϑ does not depend on the choice of the m's occuring.

Next we want to show that ϑ defines an endomorphism of $K_{n,d}^{\leq k-1}$. For this we have to show that the relations are preserved by ϑ. If we pick a relation from $(*1)$ to $(*4)$, $(\dagger 1)$ and $(\dagger 2)$ then we can find an index m distinct from the i, j, k, n's in this particular relation since we assumed $n \geq 5$. The action of ϑ is then given by conjugating every element occuring in this relation by ${f_k}^{nm}$. Since the relations form a normal subgroup this means that ϑ preserves the relations of $K_{n,d}^{\leq k-1}$ and so ϑ defines an endomorphism of this group.

Clearly, ϑ defines an automorphism of $K_{n,d}^{\leq k-1}$ for we can just define its inverse by replacing ${f_k}^{nm}$ by $(f_k^{nm})^{-1}$ in the definition of ϑ.

To obtain $K_{n,d}^{\leq k}$ from $K_{n,d}^{\leq k-1} * \langle t_k \rangle$ we only need the relations $(\dagger 1)$ and $(\dagger 2)$.

For $(\dagger 1)$ it is enough to consider all relations with $a = k$ and arbitrary b:

$$ t_k {f_b}^{ij} {t_k}^{-1} \;=\; {f_k}^{nm} {f_b}^{ij} ({f_k}^{nm})^{-1} \;=\; \vartheta({f_b}^{ij}). $$

We have to impose one relation for every $m \neq i, j$ but we have already shown above that all these elements define the same element $\vartheta({f_b}^{ij})$ of $K_{n,d}^{\leq k-1}$.

And for $a = k$ and $b < k$ the relation $(\dagger 2)$ is equivalent to

$$ \begin{aligned} t_k t_b {t_k}^{-1} \;&=\; {f_k}^{nm} \underbrace{{f_b}^{nj} ({f_k}^{nm})^{-1} ({f_b}^{nj})^{-1}}_{\text{apply } (\dagger 4)} \cdot t_b \\ &=\; {f_k}^{mn} t_b ({f_k}^{mn})^{-1} \;=\; \vartheta(t_b) \end{aligned} $$

As we have shown above this element does not depend on the choice of $m \neq n$.

Hence we have shown that

$$ K_{n,d}^{\leq k} \;\cong\; K_{n,d}^{\leq k-1} * \langle t_k \rangle / \ll t_k x {t_k}^{-1} \;=\; \vartheta(x) \;\; \forall x \in K_{n,d}^{\leq k-1} \gg $$

and this is precisely the semidirect product of $K_{n,d}^{\leq k-1}$ by $\langle t_k \rangle$. □

5.3 Affine subgroups and the construction of $\tilde{\mathcal{K}}(- ,\mathrm{n})$

We denote by $\mathfrak{F}_d$ be the free group of rank $d \geq 1$. We embed $\mathcal{K}(\mathfrak{F}_d, n)$ as usual into ${\mathfrak{F}_d}^n$, cf. Section 3.1.

Definition 5.9 *A subgroup of $\mathcal{K}(\mathfrak{F}_d,\, n)$ with $n \geq 3$ is called an* **affine subgroup** *if it is normally generated by elements of the form $(r, r^{-1}, 1, ..., 1)$, $r \in \mathfrak{F}_d$ and their $\mathfrak{S}_n$-conjugates.*

We note that for affine subgroups normal generation with respect to $\mathcal{K}(\mathfrak{F}_d, n)$ has the same effect as normal generation with respect to ${\mathfrak{F}_d}^n$: This follows since we assumed $n \geq 3$ and so we compute for $(f, 1, f^{-1}, 1, ...) \in \mathcal{K}(\mathfrak{F}_d, n)$ and $r \in \mathfrak{F}_d$:

$$\begin{array}{rlll} & (f,1,1,...,1) & (r,r^{-1},1,...,1) & (f,1,1,...,1)^{-1} \\ = & (f,1,f^{-1},...,1) & (r,r^{-1},1,...,1) & (f,1,f^{-1},...,1)^{-1}. \end{array}$$

We let G be a group and $n \geq 3$ be a natural number. We then choose a presentation $\mathfrak{F}_d/N \cong G$ of G. Then we define $R := \ll \mathcal{K}(N, n) \gg$. This is an affine subgroup of $\mathcal{K}(\mathfrak{F}_d, n)$ since it is normally generated by the elements $(s, s^{-1}, 1, ..., 1)$ with $s \in N$ and their $\mathfrak{S}_n$-conjugates. We define

$$\widetilde{\mathcal{K}}(G,\,n) \;\; := \;\; \mathcal{K}(\mathfrak{F}_d,\,n)/R.$$

Since R is $\mathfrak{S}_n$-invariant the $\mathfrak{S}_n$-action on $\mathcal{K}(\mathfrak{F}_d, n)$ descends to an action on the quotient $\widetilde{\mathcal{K}}(G, n)$ and we define

$$\widetilde{\mathcal{E}}(G,\,n) \;\; := \;\; \widetilde{\mathcal{K}}(G,\,n) \rtimes \mathfrak{S}_n.$$

with respect to this action. This is well-defined because of

Theorem 5.10 *Let $n \geq 3$ be a natural number. For every finitely generated group G the construction of $\widetilde{\mathcal{K}}(G, n)$ and its $\mathfrak{S}_n$-action do not depend on the choice of a presentation for G. Moreover, the construction of $\widetilde{\mathcal{K}}(-, n)$ is functorial in its first argument.*

If we denote by $H_2(G)$ the second group homology of G with coefficients in the integers then there is a central extension

$$0 \;\to\; H_2(G) \;\to\; \widetilde{\mathcal{K}}(G,\,n) \;\to\; \mathcal{K}(G,\,n) \;\to\; 1$$

and the image of $H_2(G)$ lies inside the commutator subgroup of $\widetilde{\mathcal{K}}(G, n)$.

PROOF. We embed $\mathcal{K}(\mathfrak{F}_d, n)$ into ${\mathfrak{F}_d}^n$. We denote by π the projection from ${\mathfrak{F}_d}^n$ onto its last $n-1$ factors. From Proposition 3.4 we know that $\ker \pi$ restricted to $\mathcal{K}(\mathfrak{F}_d, n)$ equals the commutator subgroup $[\mathfrak{F}_d, \mathfrak{F}_d]$.

We let $f \in \mathfrak{F}_d$ and $s \in N$. Then

$$[(f, 1, f^{-1}, 1, ...),\, (s, s^{-1}, 1, ...)] \;=\; ([f, s], 1, 1, 1, ...)$$

and this element lies in R. Thus $[\mathfrak{F}_d, N]$ is contained in $R \cap \ker \pi$.

Conversely, R is generated by elements of the form $(fsf^{-1}, s^{-1}, 1, ...)$ and their $\mathfrak{S}_n$-conjugates where f runs through $\mathfrak{F}_d$ and s runs through N. From this it follows that every element of R can be written as a product of the form

$$\prod_i (f_i s_i {f_i}^{-1}, 1, ..., {s_i}^{-1}, 1, ...) = \prod_i \big(([f_i, s_i], 1, ...) \cdot (s_i, 1, ..., {s_i}^{-1}, 1, ...)\big)\,.$$

with $f_i \in \mathfrak{F}_d$ and $s_i \in N$. Using that $[\mathfrak{F}_d, N]$ is a normal subgroup of $\mathfrak{F}_d$ we see that every element of R can be written as a product

$$\left(\prod_i ([f_i', s_i'], 1, ...)\right) \cdot \left(\prod_j (s_j', 1, ..., s_j'^{-1}, 1, ...)\right).$$

Such an element lies in $\ker \pi$ if and only if the second product over the j's lies in $\ker \pi$. From Proposition 3.4 applied to $\mathcal{K}(N, n)$ we see that such an element is of the form $(s', 1, ..., 1)$ with $s' \in [N, N]$. In particular, an element of $\ker \pi \cap R$ is a product of elements $(r, 1, ..., 1)$ with $r \in [\mathfrak{F}_d, N]$. Thus we have shown that $R \cap \ker \pi$ is equal to $[\mathfrak{F}_d, N]$.

So there is the following diagram of groups with exact rows and where the maps downwards are injective:

$$\begin{array}{ccccccccc} 1 & \to & [\mathfrak{F}_d, N] & \to & \overbrace{\ll \mathcal{K}(N, n) \gg}^{=R} & \to & N^{n-1} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N \cap [\mathfrak{F}_d, \mathfrak{F}_d] & \to & N^n \cap \mathcal{K}(\mathfrak{F}_d, n) & \to & N^{n-1} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & [\mathfrak{F}_d, \mathfrak{F}_d] & \to & \mathcal{K}(\mathfrak{F}_d, n) & \xrightarrow{\pi} & \mathfrak{F}_d{}^{n-1} & \to & 1 \end{array}$$

Taking successive quotients we exhibit the group $\mathcal{K}(\mathfrak{F}_d, n)/R$ as an extension of $(N \cap [\mathfrak{F}_d, \mathfrak{F}_d])/[\mathfrak{F}_d, N]$ by $\mathcal{K}(\mathfrak{F}_d, n)/(N^n \cap \mathcal{K}(\mathfrak{F}_d, n))$. The latter group is isomorphic to $\mathcal{K}(G, n)$ whereas the first group is isomorphic to $H_2(G)$ by Hopf's theorem (quoted as Theorem 5.23). Hence we get an extension

$$1 \to H_2(G) \to \mathcal{K}(\mathfrak{F}_d, n)/R \to \mathcal{K}(G, n) \to 1.$$

We can also take the quotient of the upper exact row by the lower exact row and obtain the following short exact sequence (cf. Corollary 5.14)

$$1 \to [\mathfrak{F}_d, \mathfrak{F}_d]/[\mathfrak{F}_d, N] \to \mathcal{K}(\mathfrak{F}_d, n)/R \to G^{n-1} \to 1.$$

The inclusion of $H_2(G)$ into $\mathcal{K}(\mathfrak{F}_d, n)/R$ is given by

$$H_2(G) = (N \cap [\mathfrak{F}_d, \mathfrak{F}_d])/[\mathfrak{F}_d, N] \hookrightarrow [\mathfrak{F}_d, \mathfrak{F}_d]/[\mathfrak{F}_d, N] \hookrightarrow \mathcal{K}(\mathfrak{F}_d, n)/R.$$

Every element of the group in the middle can be written as product of commutators

$$[(f_1, f_1{}^{-1}, 1, ...), (f_2, 1, f_2{}^{-1}, ...)]$$

where the f_i's are appropriate lifts to $\mathfrak{F}_d$. Hence this group lies in the commutator subgroup of $\mathcal{K}(\mathfrak{F}_d, n)/R$. Since $H_2(G)$ is a subgroup of this group in the middle also $H_2(G)$ lies in the commutator subgroup of $\mathcal{K}(\mathfrak{F}_d, n)/R$.

Every element of $H_2(G)$ maps to an element in $\mathcal{K}(\mathfrak{F}_d, n)/R$ that can be lifted to an element of $\mathcal{K}(\mathfrak{F}_d, n)$ of the form $\vec{x} = (x, 1, ..., 1)$ with $x \in N \cap [\mathfrak{F}_d, \mathfrak{F}_d]$. Given any element $\vec{y} = (y_1, ..., y_n)$ of $\mathcal{K}(\mathfrak{F}_d, n)$ we compute

$$\begin{aligned} \vec{y}\vec{x}\vec{y}^{-1} &= (y_1 x y_1{}^{-1}, 1, ..., 1) \\ &= (\underbrace{[y_1,\, x]}_{\in[\mathfrak{F}_d, N]}, 1, ..., 1) \cdot (x, 1, ..., 1) \\ &\equiv \vec{x} \mod [\mathfrak{F}_d, N]. \end{aligned}$$

Hence $H_2(G)$ maps into the centre of $\mathcal{K}(\mathfrak{F}_d, n)/R$.

Suppose we are given two free groups $\mathfrak{F}_d$ and $\mathfrak{F}_{d'}$, two normal subgroups N and N' in them and a homomorphism α between their quotients. Since free groups are projective objects there exists a homomorphism $\varphi : \mathfrak{F}_d \to \mathfrak{F}_{d'}$ making the following diagram commute

$$\begin{array}{ccc} \mathfrak{F}_d & \overset{\exists\varphi}{\dashrightarrow} & \mathfrak{F}_{d'} \\ \downarrow & & \downarrow \\ \mathfrak{F}_d/N & \overset{\alpha}{\to} & \mathfrak{F}_{d'}/N'. \end{array}$$

Then φ induces a map from N to N' and hence a map from $\mathcal{K}(N, n)$ to $\mathcal{K}(N', n)$. We will call R the normal closure of $\mathcal{K}(N, n)$ in ${\mathfrak{F}_d}^n$ and similarly R' the normal closure of $\mathcal{K}(N', n)$ in ${\mathfrak{F}_{d'}}^n$. Then φ induces a map from R to R' and we get an induced homomorphism

$$\varphi \;\; : \;\; \mathcal{K}(\mathfrak{F}_d,\, n)/R \;\; \to \;\; \mathcal{K}(\mathfrak{F}_{d'},\, n)/R'.$$

We want to show that the map induced by φ does not depend on the choice of the lift of α. So suppose we have a second map $\varphi' : \mathfrak{F}_d \to \mathfrak{F}_{d'}$ lifting α. Since elements of the form $(f, f^{-1}, 1, ..., 1)$ generate $\mathcal{K}(\mathfrak{F}_d, n)$ it is enough to compare the induced morphisms on these elements. For $f \in \mathfrak{F}_d$ there exists an element $s_f \in N'$ (depending on f) such that $\varphi(f) = \varphi'(f)s_f$. Hence

$$\begin{aligned} \varphi'((f, f^{-1}, 1, ..., 1)) &= (\varphi'(f), \varphi'(f)^{-1}, 1, ..., 1) \\ &= (\varphi(f)s_f, {s_f}^{-1}\varphi(f)^{-1}, 1, ..., 1) \\ &= \varphi(f) \cdot \underbrace{(s_f, \varphi(f){s_f}^{-1}\varphi(f), 1, ..., 1)}_{\in R'}. \end{aligned}$$

So the induced maps coincide. In particular, if $\mathfrak{F}_d = \mathfrak{F}_{d'}$, $N = N'$ and α is the identity we can choose φ to be the identity. By the uniqueness just shown we see that the identity induces the identity.

If α is an isomorphism from $\mathfrak{F}_d/N$ to $\mathfrak{F}_{d'}/N'$ then the induced homomorphism from $\mathcal{K}(\mathfrak{F}_d, n)/R$ to $\mathcal{K}(\mathfrak{F}_{d'}, n)/R'$ must be an isomorphism. This shows that this quotient does not depend on the choice of the presentation and we may refer to both quotients as $\widetilde{\mathcal{K}}(G, n)$.

Since R is $\mathfrak{S}_n$-invariant the action of $\mathfrak{S}_n$ on $\mathcal{K}(\mathfrak{F}_d, n)$ descends to the quotient $\mathcal{K}(\mathfrak{F}_d, n)/R$. A similar reasoning as above shows that also this action only depends on G and n. $\square$

Again we denote by p_1 the projection from ${\mathfrak{F}_d}^n$ onto its first factor. By abuse of notation we will also denote its restriction to $\mathcal{K}(\mathfrak{F}_d,\, n)$ with p_1. As a consequence of the previous theorem we can determine quotients by affine subgroups:

Corollary 5.11 *Suppose we are a given a natural number $n \geq 3$ and an affine subgroup R of $\mathcal{K}(\mathfrak{F}_d, n)$. We define*

$$N \;:=\; p_1(R) \quad \textit{and} \quad G \;:=\; \mathfrak{F}_d/N.$$

Then there is an isomorphism

$$\mathcal{K}(\mathfrak{F}_d,\, n)/R \;\cong\; \widetilde{\mathcal{K}}(G,\, n).$$

In particular, the quotient is completely determined by G and n.

PROOF. Since p_1 is surjective the subgroup N of $\mathfrak{F}_d$ is indeed normal. Also R is stable under $\mathfrak{S}_n$ and so N does not depend on the projection we have chosen.

We have a short exact sequence

$$1 \;\to\; N^n \cap \mathcal{K}(\mathfrak{F}_d,\, n) \;\to\; \mathcal{K}(\mathfrak{F}_d,\, n) \;\to\; \mathcal{K}(G,\, n) \;\to\; 1.$$

Clearly $\mathcal{K}(N, n)$ is a subgroup of R and since R is a normal subgroup also its normal closure with respect to $\mathcal{K}(\mathfrak{F}_d, n)$ is contained in R. Conversely, R is normally generated by elements of the form $(r, r^{-1}, 1, ...)$ and their $\mathfrak{S}_n$-conjugates. Since these r's lie in N we conclude that R must be contained in $\ll \mathcal{K}(N, n) \gg$ and so R and $\ll \mathcal{K}(N, n) \gg$ coincide. Hence $\mathcal{K}(\mathfrak{F}_d, n)/R$ is isomorphic to $\widetilde{\mathcal{K}}(G, n)$ by definition of the latter group. $\square$

Corollary 5.12 *If $\alpha : G \to H$ is a homomorphism between finitely generated groups then there are induced maps*

$$\begin{array}{ccccccccc} 0 & \to & H_2(G) & \to & \widetilde{\mathcal{K}}(G,\, n) & \to & \mathcal{K}(G,\, n) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_2(H) & \to & \widetilde{\mathcal{K}}(H,\, n) & \to & \mathcal{K}(H,\, n) & \to & 1 \end{array} .$$

The induced map $\mathcal{K}(G, n) \to \mathcal{K}(H, n)$ coincides with the one induced from $\mathcal{K}(-, n)$. The map from $H_2(G)$ to $H_2(H)$ can be made compatible with the map induced from group homology.

PROOF. We let $\mathfrak{F}_d/N \cong G$ and $\mathfrak{F}_{d'}/N' \cong H$ be presentations of G and H, respectively. Again we lift $\alpha : G \to H$ to a map $\varphi : \mathfrak{F}_d \to \mathfrak{F}_{d'}$. The map between the two H_2's is the one induced from φ and

$$\begin{array}{ccc} H_2(G) & \cong & (N \cap [\mathfrak{F}_d, \mathfrak{F}_d])/[\mathfrak{F}_d, N] \\ \downarrow & & \downarrow \\ H_2(H) & \cong & (N' \cap [\mathfrak{F}_{d'}, \mathfrak{F}_{d'}])/[\mathfrak{F}_{d'}, N'] \end{array}$$

By [Br, Exercise II.6.3.b] this can be made compatible with the homomorphism $\alpha_* : H_2(G) \to H_2(H)$ on homology. □

The connection with the universality results for $\mathcal{K}(-, n)$ given in Proposition 3.9 and Corollary 3.10 is as follows:

Corollary 5.13 *Let $n \geq 3$ be a natural number and G be a finitely generated group. With respect to the action of $\mathfrak{S}_n$ on $\widetilde{\mathcal{K}}(G, n)$ given by Theorem 5.10 we define*

$$X \;:=\; \widetilde{\mathcal{K}}(G, n) \quad \textit{and} \quad Y \;:=\; X_{\mathfrak{S}_n}/X_{\mathfrak{S}^{(1)}_{n-1}}.$$

Then Y is isomorphic to G and X is equal to $X_{\mathfrak{S}_n}$. The universal homomorphism given by Proposition 3.9 takes the following form:

$$\begin{array}{ccccccccc} 1 & \to & \bigcap_{i=1}^n X_{\mathfrak{S}^{(i)}_{n-1}} & \to & X_{\mathfrak{S}_n} & \to & \mathcal{K}(Y, n) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_2(G) & \to & \widetilde{\mathcal{K}}(G, n) & \to & \mathcal{K}(G, n) & \to & 1 \end{array}$$

where the maps downwards are isomorphisms.

PROOF. Let $\mathfrak{F}_d/N$ be a presentation of G. Since $[\mathcal{K}(\mathfrak{F}_d, n), \mathfrak{S}_n]$ equals $\mathcal{K}(\mathfrak{F}_d, n)$ the same is true for the quotient by the affine subgroup R. Hence we have $[X, \mathfrak{S}_n] = X$. Also, identifying $[\mathcal{K}(\mathfrak{F}_d, n), \mathfrak{S}^{(1)}_{n-1}]$ with $\mathcal{K}(\mathfrak{F}_d, n-1)$ we conclude that $[X, \mathfrak{S}^{(1)}_{n-1}]$ is the same as $\widetilde{\mathcal{K}}(G, n-1)$. Using the exact sequence of the statement of Theorem 5.10 we conclude

$$Y \overset{\text{def}}{=} X_{\mathfrak{S}_n}/X_{\mathfrak{S}^{(1)}_{n-1}} = \mathcal{K}(G, n)/\mathcal{K}(G, n-1) \cong G.$$

Applying Proposition 3.9 we get our statement. □

Corollary 5.14 *Let $n \geq 3$ and G be a finitely generated group. We choose a presentation $\mathfrak{F}_d/N \cong G$ of G. Then there exists a short exact sequence*

$$1 \;\to\; [\mathfrak{F}_d, \mathfrak{F}_d]/[\mathfrak{F}_d, N] \;\to\; \widetilde{\mathcal{K}}(G, n) \;\to\; G^{n-1} \;\to\; 1.$$

If G is perfect then the group on the left is just its universal central extension.

PROOF. We have seen the above short exact sequence already in the proof of Theorem 5.10. For details on universal central extensions of perfect groups we refer to Section 5.5. □

Corollary 5.15 *Let $n \geq 3$ and consider the following properties of groups:*

finite, nilpotent, perfect, solvable.

Then G has one of the properties above if and only if $\widetilde{\mathcal{K}}(G,n)$ has the respective property.

PROOF. If G is finite then so are $H_2(G)$ and $\mathcal{K}(G,n)$. Hence if G is finite then so is $\widetilde{\mathcal{K}}(G,n)$ being an extension of two finite groups.

Since $H_2(G)$ is an abelian group it is also solvable and even nilpotent. Hence if G is solvable (resp. nilpotent) then so is $\widetilde{\mathcal{K}}(G,n)$ being a (central) extension of two solvable (resp. nilpotent) groups.

If G is perfect then so is its universal central extension $\tilde{G}$. So in this case also $\widetilde{\mathcal{K}}(G,n)$ is perfect being an extension of two perfect groups by Corollary 5.14.

If $\widetilde{\mathcal{K}}(G,n)$ is finite (resp. nilpotent, perfect, solvable) then so is G being a quotient of $\widetilde{\mathcal{K}}(G,n)$. □

Remark 5.16 *Since $H_2(G)$ occurs as a subgroup of the commutator subgroup of $\widetilde{\mathcal{K}}(G,n)$ it follows that $\widetilde{\mathcal{K}}(G,n)$ cannot be abelian if $H_2(G)$ is non-trivial. For example, if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ then $H_2(G) = \mathbb{Z}_2$ and so $\widetilde{\mathcal{K}}(G,n)$ is non-abelian.*

Despite the complexity of $\widetilde{\mathcal{K}}(-,n)$ we can always compute its abelianisation:

Corollary 5.17 *Let $n \geq 3$ and let G be a finitely generated group. Then there are isomorphisms*

$$\widetilde{\mathcal{K}}(G,n)^{\mathrm{ab}} \;\cong\; \widetilde{\mathcal{K}}(G^{\mathrm{ab}},n)^{\mathrm{ab}} \;\cong\; \mathcal{K}(G,n)^{\mathrm{ab}} \;\cong\; \mathcal{K}(G^{\mathrm{ab}},n) \;\cong\; (G^{\mathrm{ab}})^{n-1}.$$

We note that only the last isomorphism is not natural.

PROOF. We consider the short exact sequence of Theorem 5.10. Since $H_2(G)$ lies inside the commutator subgroup of $\widetilde{\mathcal{K}}(G,n)$ the induced homomorphism of abelianisation

$$\widetilde{\mathcal{K}}(G,n)^{\mathrm{ab}} \;\;\to\;\; \mathcal{K}(G,n)^{\mathrm{ab}}$$

is an isomorphism. The abelianisation of $\mathcal{K}(G,n)$ is computed in Proposition 3.8 and induces an isomorphism of this group with $\mathcal{K}(G^{\mathrm{ab}},n)$ which is isomorphic to $(G^{\mathrm{ab}})^{n-1}$ by Corollary 3.5.

By what we have just proved $\widetilde{\mathcal{K}}(G^{\mathrm{ab}},n)^{\mathrm{ab}}$ is isomorphic to $\mathcal{K}(G^{\mathrm{ab}},n)$ proving the remaining isomorphism. □

5.4 Examples

We now compute $\widetilde{\mathcal{K}}(-,n)$ in some cases. Since the computation of H_2 of a group is a difficult business we will only give a couple of examples that will be important in the sequel.

Example 5.18 *Let $n \geq 3$. If G is a (possibly infinite) cyclic group then there are isomorphisms*

$$\widetilde{\mathcal{K}}(G,n) \;\cong\; \mathcal{K}(G,n) \;\cong\; G^{n-1}.$$

PROOF. Hopf's theorem (Theorem 5.23) shows us that $H_2(-,\mathbb{Z})$ vanishes for cyclic groups. After applying Corollary 3.5 we are done. □

Example 5.19 *Let $n \geq 3$. Then there is an isomorphism*

$$H_2(\mathbb{Z}^d) \;\cong\; \mathbb{Z}^{d(d-1)/2}$$

and we get a central extension

$$0 \;\to\; \mathbb{Z}^{d(d-1)/2} \;\to\; \widetilde{\mathcal{K}}(\mathbb{Z}^d,\, n) \;\to\; \mathbb{Z}^{d(n-1)} \;\to\; 1$$

However, Remark 5.16 tells us that $\widetilde{\mathcal{K}}(\mathbb{Z}^d, n)$ cannot be abelian for $d \geq 2$.

PROOF. We consider the d-dimensional torus $T_d := S^1 \times ... \times S^1$ in the sense of algebraic topology. Applying Theorem 5.22 to T_d we conclude that $H_2(\mathbb{Z}^d)$ is isomorphic to $H_2(T_d,\, \mathbb{Z})$ which is isomorphic to $\mathbb{Z}^{d(d-1)/2}$. We compute $\mathcal{K}(\mathbb{Z}^d, n)$ via Corollary 3.5 and apply Theorem 5.10. □

Example 5.20 *Let $n \geq 3$. We let Π_g be the fundamental group of a smooth projective algebraic curve of genus $g \geq 1$, cf. Section 1.1. Then there exists an isomorphism*

$$H_2(\Pi_g) \;\cong\; \mathbb{Z}$$

and we get a central extension

$$0 \;\to\; \mathbb{Z} \;\to\; \widetilde{\mathcal{K}}(\Pi_g,\, n) \;\to\; \mathcal{K}(\Pi_g,\, n) \;\to\; 1.$$

PROOF. We forget the complex structure and consider a smooth projective curve only as a closed orientable surface S_g of genus $g \geq 1$. This is a $K(\Pi_g, 1)$-space and we can apply Theorem 5.22 to conclude that $H_2(\Pi_g)$ is isomorphic to $H_2(S_g,\, \mathbb{Z})$ which is isomorphic to $\mathbb{Z}$. The rest follows from Theorem 5.10. □

Skipping through the references given at the beginning of Section 5.5 we find the following

Examples 5.21 *The following H_2's vanish*

$$H_2(\mathbb{Z}_n) \;=\; H_2(Q_8) \;=\; H_2(\mathbb{Z}) \;=\; H_2(D_\infty) \;=\; 1,$$

where Q_8 denotes the quaternion group and D_∞ denotes the infinite dihedral group. For the dihedral groups of order $2n$ we have

$$H_2(D_{2n}) \;=\; \begin{cases} 1 & \text{if } n \text{ is odd} \\ \mathbb{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$

For $n \geq 4$ it is known that

$$H_2(\mathbb{Z}_2 \times \mathbb{Z}_2) \;=\; H_2(\mathfrak{S}_n) \;=\; \mathbb{Z}_2.$$

Appendix to Section 5

5.5 Group homology and the computation of H_2

In this section we first recall the construction of group homology. Then we give some of its properties and give some statements that allow us to actually compute H_2 of a given group. As references we refer to [Br, Chapter II], [We, Chapter 6], [Rot, Chapter 7] and [Rot, Chapter 11].

Let G be an arbitrary group. For a left G-module M we define its *module of co-invariants* to be the quotient of M by the module I_G generated by all elements $g \cdot m - m$ for all $g \in G$ and $m \in M$:

$$M_G \;\;:=\;\; M/I_G.$$

Taking co-invariants defines a right exact functor for left G-modules and we can consider its left derived functor. We define the *i.th homology $H_i(G)$* of G to be the i.th left derived functor of $-_G$ applied to the G-module $\mathbb{Z}$ with trivial G-action:

$$H_i(G) \;\;:=\;\; H_i(G, \mathbb{Z}).$$

Using the standard resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}[G]$ it is not hard to prove that for all groups

$$\begin{array}{rcl} H_0(G) & \cong & \mathbb{Z} \\ H_1(G) & \cong & G^{\mathrm{ab}} \end{array}$$

holds true. Clearly, all homology groups are abelian groups. Using again the standard resolution mentioned before one can show that if G is a finite group then also its homology groups are finite.

The origins of group homology lie in algebraic topology: We recall that a connected CW-complex Y is called a $K(G,1)$-complex if $\pi_1^{\mathrm{top}}(Y) \cong G$ and if its universal cover is contractible.

Theorem 5.22 *For a $K(G,1)$-complex Y there exist for all $i \geq 0$ isomorphisms*

$$H_i(G) \;\cong\; H_i(Y, \mathbb{Z})$$

where $H_i(Y, \mathbb{Z})$ denotes the singular homology of the topological space Y.

A short exact sequence

$$0 \;\rightarrow\; A \;\rightarrow\; X \;\rightarrow\; G \;\rightarrow\; 1$$

is called a *central extension* of G if A lies in the centre of X. A central extension $0 \rightarrow A \rightarrow X \rightarrow G \rightarrow 1$ is called a *universal central extension* if for every central extension $1 \rightarrow B \rightarrow Y \rightarrow G \rightarrow 1$ there exists a unique homomorphism from X to Y making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \rightarrow & X & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & || & & \\ 0 & \rightarrow & B & \rightarrow & Y & \rightarrow & G & \rightarrow & 1 \end{array}$$

If such a universal central extension exists it is unique up to isomorphism.

Central extensions $0 \rightarrow A \rightarrow X \rightarrow G \rightarrow 1$ with a fixed abelian group A are classified by $\mathrm{Hom}(H_2(G), A)$. In particular, central extensions with $\mathbb{C}^*$ are classified by $\mathrm{Hom}(H_2(G), \mathbb{C}^*) \cong H^2(G, \mathbb{C}^*) =: M(G)$. This latter group is called the *Schur multiplier* of G. If G is finite then Pontryagin duality provides us with a non-canonical isomorphism between $H_2(G)$ and $M(G)$.

A group G has a universal central extension $\tilde{G}$ if and only if it is perfect. In this case the universal extension takes the form

$$0 \;\rightarrow\; H_2(G) \;\rightarrow\; \tilde{G} \;\rightarrow\; G \;\rightarrow\; 1.$$

Now let N be a normal subgroup of a free group F such that $G \cong F/N$. Then there is a central extension

$$0 \rightarrow (N \cap [F,F])/[N,F] \rightarrow [F,F]/[N,F] \rightarrow [G,G] \rightarrow 1.$$

In case G is a perfect group this is exactly its universal central extension. But even in the case where G is not necessarily perfect we have the following

Theorem 5.23 (Hopf) *Let G be an arbitrary group. If N is a normal subgroup of a free group F such that $G \cong F/N$ then*

$$H_2(G) \;\cong\; (N \cap [F,F])/[F,N].$$

5.6 Examples from the theory of Coxeter groups

There is a certain "Coxeter flavour" in connection with the groups $\mathcal{E}(G, n)$ as for example Lemma 5.5 indicates. For the following we refer to [Hum, Chapter 5].

A symmetric $n \times n$ matrix $M = (m_{ij})_{i,j}$ with entries $m_{ij} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ is called a *Coxeter matrix* if

1. $m_{ii} = 1$ for all i and
2. $m_{ij} \geq 2$ for all $i \neq j$.

Let $S = \{s_1, ..., s_n\}$ be a set with n elements and $M = (m_{ij})_{i,j}$ a $n \times n$ Coxeter matrix. A group given by generators and relations

$$W(S, M) \quad := \quad \langle s_i \in S \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle$$

is called a *Coxeter group*. The associated *Coxeter graph* is defined to be the (undirected) graph with

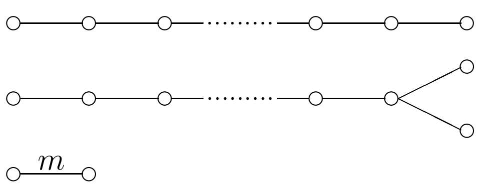
vertices	:	the elements of S
edges	:	there is an edge joining s_i to s_j if and only if $m_{ij} \geq 3$.

If $m_{ij} \geq 4$ then we will write this number above the edge joining s_i to s_j.

The finite Coxeter groups are classified, cf. [Hum, Chapter I.2].

We now consider the following three series of finite Coxeter groups given by the following graphs:

A_n ; $n \geq 1$
D_n ; $n \geq 4$
$\mathsf{I}_2(m)$



It is known that

$$W(\mathsf{A}_{n-1}) \quad := \quad \langle \tau_k \,|\, {\tau_k}^2,\, (\tau_k \tau_{k+1})^3,\, (\tau_k \tau_j)^2 \text{ for } |k - j| \geq 2 \rangle$$

is isomorphic to $\mathfrak{S}_n$ by sending τ_k to the transposition $(k\, k+1)$.

The upper chain forms a subgraph of type A_{n-1} inside D_n. This defines a subgroup isomorphic to $\mathfrak{S}_n$ inside $W(\mathsf{D}_n)$. We define a split surjection

$$\psi \;:\; W(\mathsf{D}_n) \;\twoheadrightarrow\; \mathfrak{S}_n$$

being the identity when restricted to the subgroup $\mathfrak{S}_n$ and sending the remaining reflection to the image of the reflection "lying above" it in the graph D_n. From the description in [Hum, Chapter 2.10] we get

Example 5.24 *The homomorphism ψ makes $\ker\psi$ into $\mathcal{K}(\mathbb{Z}_2, n)$ and induces an isomorphism*

$$W(\mathsf{D}_n) \;\cong\; \mathcal{E}(\mathbb{Z}_2,\, n).$$

Next we define a split surjection

$$\psi \;:\; W(\mathsf{I}_2(m)) \;\twoheadrightarrow\; \mathfrak{S}_2 \cong \mathbb{Z}_2$$

by sending both reflections to the non trivial element of $\mathbb{Z}_2$. This Coxeter group is the dihedral group of order $2m$.

Example 5.25 *If m is odd then ψ makes $\ker\psi$ into $\mathcal{K}(\mathbb{Z}_m, n)$ and induces an isomorphism*

$$W(\mathsf{I}_2(m)) \;\cong\; \mathcal{E}(\mathbb{Z}_m,\, n).$$

If m is even then

$$W(\mathsf{I}_2(m))_{\mathfrak{S}_2} \;\cong\; W(\mathsf{I}_2(\tfrac{m}{2})).$$

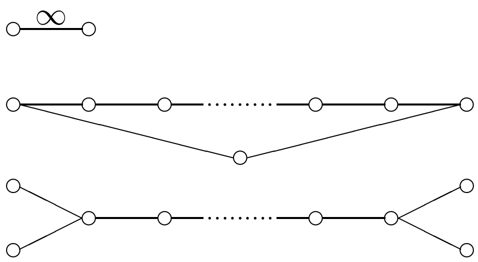
The effect of the previous example is best explained by the fact that there are roots of different lengths that may or may not be conjugate to short roots. This is why we are only interested in simply laced graphs.

Next we consider the following graphs giving rise to infinite Coxeter groups (they are examples of affine Weyl groups):

$\tilde{\mathsf{A}}_1$

$\tilde{\mathsf{A}}_n$ $\quad n \geq 2$

$\tilde{\mathsf{D}}_n$ $\quad n \geq 4$



The upper chain forms a subgraph of type A_n inside $\tilde{\mathsf{A}}_n$. We number it from the left to the right by τ_1,...,τ_n. This defines a subgroup isomorphic to $\mathfrak{S}_{n+1}$ inside $W(\tilde{\mathsf{A}}_n)$ where we identify τ_k with the transposition $(k\,k+1)$. Again, we may define a split surjection

$$\psi \;:\; W(\tilde{\mathsf{A}}_n) \;\twoheadrightarrow\; \mathfrak{S}_{n+1}$$

by sending the “extra“ reflection to $(1\,n)$. We refer to [Hum, Chapter 4.2] for

Example 5.26 *The homomorphism ψ makes $\ker\psi$ into $\mathcal{K}(\mathbb{Z}, n)$ and induces an isomorphism*

$$W(\tilde{\mathsf{A}}_n) \;\cong\; \mathcal{E}(\mathbb{Z},\, n+1).$$

Under the isomorphism given in the example the “extra“ reflection maps to the element $(1, 0, ..., 0, -1)(1\, n)$ of $\mathcal{E}(\mathbb{Z}, n)$

The upper chain forms a subgraph of type A_{n-1} inside $\tilde{\mathsf{D}}_n$. This defines a subgroup isomorphic to $\mathfrak{S}_n$ inside $W(\tilde{\mathsf{D}}_n)$. We define a split surjection

$$\psi \;:\; W(\tilde{\mathsf{D}}_n) \;\twoheadrightarrow\; \mathfrak{S}_n$$

being the identity when restricted to the subgroup $\mathfrak{S}_n$ and sending a remaining reflection to the image of the respective reflection “lying above“ it in the graph $\tilde{\mathsf{D}}_n$. We leave it to the reader to show that we get the

Example 5.27 *The homomorphism ψ makes* $\ker\psi$ *into $\mathcal{K}(D_\infty, n)$ and induces an isomorphism*

$$W(\tilde{\mathsf{D}}_n) \;\cong\; \mathcal{E}(D_\infty, n)$$

where D_∞ denotes the infinite dihedral group.

6 Conclusion

Jetzt nehmt den Wein! Jetzt ist es Zeit, Genossen!
Leert eure gold'nen Becher zu Grund!
Dunkel ist das Leben, ist der Tod!

6.1 The algorithm of Zariski and van Kampen

Let C be a reduced but not necessarily smooth or irreducible projective curve of degree d in the complex projective plane. We choose a generic line $\tilde{\ell} \subset \mathbb{P}^2$, i.e. a line that intersects C in d distinct points. We set $\mathbb{A}^2 := \mathbb{P}^2 - \tilde{\ell}$ and denote the intersection $C \cap \mathbb{A}^2$ again by C. We are interested in computing the fundamental groups

$$\pi_1^{\text{top}}(\mathbb{P}^2 - C) \quad \text{and} \quad \pi_1^{\text{top}}(\mathbb{A}^2 - C).$$

An algorithm that yields presentations of these groups is given in van Kampen's article [vK]. The result was known to Zariski before and also Enriques, Lefschetz and Picard should be mentioned in this context.

We now follow [Ch] and [Mo] to describe this algorithm: We choose a generic line ℓ in $\mathbb{A}^2$, i.e. a line intersecting C in d distinct points. The inclusion maps induce group homomorphisms

$$\begin{array}{rcl} \pi_1^{\text{top}}(\mathbb{A}^2 - C) & \to & \pi_1^{\text{top}}(\mathbb{P}^2 - C) \\ \pi_1^{\text{top}}(\ell - \ell \cap C) & \to & \pi_1^{\text{top}}(\mathbb{A}^2 - C). \end{array}$$

Both homomorphisms are surjective. A modern proof for this is for example given by [N, Proposition 2.1] and its corollaries.

The underlying topological space of $\ell - \ell \cap C$ can be identified with $\mathbb{R}^2$ with d points cut out. Hence its fundamental group is the free group of rank d. To get a system of d generators we may proceed as follows: We let u_0 be the base point for the fundamental group of $\ell - \ell \cap C$. We let w_1,...,w_d be the points of $\ell \cap C$. Next we choose paths γ_i from u_0 to w_i for all $i = 1, ..., d$ and assume that distinct γ_i's meet only in u_0. Next we shorten the γ_i's such that they stop before reaching their w_i's. Putting a little circle around w_i at the end of the so shortened γ_i's we obtain loops Γ_i that lie in $\ell - \ell \cap C$. Loops like this are usually called *simple loops* and we already met them in Section 4.4. These Γ_i's freely generate the fundamental group of $\ell - \ell \cap C$:

$$\pi_1^{\text{top}}(\ell - \ell \cap C,\, u_0) \;=\; \langle \Gamma_i,\, i = 1, ..., d \rangle \;\cong\; \mathfrak{F}_d.$$

We consider the closure $\bar{\ell}$ of ℓ inside $\mathbb{P}^2$ and denote by $\infty := \bar{\ell} - \ell$ the point at infinity. We may put an orientation on the Γ_i's and order them in such a way that the composition

$$\delta \ := \ \Gamma_1 \cdot \ldots \cdot \Gamma_d$$

is homotopic to a loop around ∞. Sticking to Moishezon's terminology we will call such a system $\{\Gamma_i\}_{i=1,\ldots,d}$ a *good ordered system of generators*. By abuse of notation we will denote the image of Γ_i in $\pi_1^{\text{top}}(\mathbb{A}^2 - C)$ and $\pi_1^{\text{top}}(\mathbb{P}^2 - C)$ again by Γ_i.

Having established a generating set we have to determine all the relations that hold between them: For this, we choose yet another line ℓ' inside $\mathbb{A}^2$ intersecting C in d distinct points. We denote the projection from ∞ to ℓ' by π. We will call a point in the fibre of π *exceptional* if it lies on a singularity of C or if the fibre is tangent to C at this point. If we choose the lines ℓ and ℓ' generically then there is only a finite set M of points on ℓ' such that there are exceptional points in the fibre $\pi^{-1}(m)$ if and only if $m \in M$. Moreover, since we have assumed that ℓ and ℓ' are generic there is at most one exceptional point in each fibre of π. Furthermore, we may assume that all tangent points are simple, i.e. have multiplicity two. We will also assume that ℓ and ℓ' intersect in u_0 so that we can take this point as the base point for all fundamental groups involved.

The map π restricted to $E := \mathbb{A}^2 - C - \bigcup_{m\in M} \pi^{-1}(m)$ is a $\mathcal{C}^\infty$ fibre bundle with base $\ell' - M$ and fibre $\ell - \ell \cap C$. Since the homotopy type of $\ell' - M$ is a wedge of 1-spheres its second homotopy group vanishes. The long exact sequence of homotopy groups of a fibration then becomes a short exact sequence of fundamental groups

$$1 \to \pi_1^{\text{top}}(\ell - \ell\cap C) \to \pi_1^{\text{top}}(\mathbb{A}^2 - C - \bigcup_{m\in M} \pi^{-1}(m)) \to \pi_1^{\text{top}}(\ell' - M) \to 1.$$

As above we may construct loops based at u_0 that form a good ordered system of generators for $\pi_1^{\text{top}}(\ell' - M)$. These loops also lie in E and give elements in $\pi_1^{\text{top}}(E)$ that lift the system of generators of $\pi_1^{\text{top}}(\ell' - M)$. Since this system of generators generates $\pi_1^{\text{top}}(\ell' - M)$ freely this lift extends to a homomorphism s and we can split the short exact sequence above.

Using the natural inclusion maps of spaces we see that the surjection from $\pi_1^{\text{top}}(\ell - \ell\cap C)$ to $\pi_1^{\text{top}}(\mathbb{A}^2 - C)$ factors over $\pi_1^{\text{top}}(E)$. Hence we have a surjective homomorphism

$$\pi_1^{\text{top}}(\mathbb{A}^2 - C - \textstyle\bigcup_{m\in M} \pi^{-1}(m)) \ \twoheadrightarrow \ \pi_1^{\text{top}}(\mathbb{A}^2 - C).$$

It is clear that $s(\pi_1^{\text{top}}(\ell' - M))$ lies in the kernel of this map. The main point is that the kernel is exactly the group normally generated by $s(\pi_1^{\text{top}}(\ell' - M))$ inside $\pi_1^{\text{top}}(E)$. For a proof of this in our setup we refer to [Ch, Partie 3.2].

This can be also formulated as follows: Using the splitting s we can define the *monodromy* homomorphism

$$\begin{array}{ccccc} \vartheta & : & \pi_1^{\mathrm{top}}(\ell' - M) & \to & \mathrm{Aut}(\pi_1^{\mathrm{top}}(\ell - \ell \cap C)) \\ & & \gamma & \mapsto & (\Gamma \mapsto s(\gamma) \cdot \Gamma \cdot s(\gamma)^{-1}). \end{array}$$

Thus $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - C)$ is generated by the Γ_i's $i = 1, ..., d$ subject to the relations $\Gamma_i = \vartheta(\gamma)(\Gamma_i)$ for all i's. Of course, it is enough if γ runs through a generating set of $\pi_1^{\mathrm{top}}(\ell' - M)$ e.g. a good ordered system of generators. This provides us with a finite presentation of the fundamental group we are looking for. Taking the quotient of the subgroup normally generated by δ inside $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - C)$ we obtain $\pi_1^{\mathrm{top}}(\mathbb{P}^2 - C)$.

We now compute locally the monodromies that are interesting for us: For this we let z and w be coordinates on $\mathbb{A}^2$ and define the lines $\ell' := \{z = z_0\}$ and $\ell := \{w = w_0\}$ with $w_0 \geq 2$ and $z_0 \leq -2$. We assume that the projection π is given by $(z, w) \mapsto w$.

In case of a simple tangent point we may assume that C is given by the equation $z^2 = w$. The fibre of π consists of two points except for $w = 0$, $z = z_0$ hence this is the only point of M. We let γ be a simple loop around this point in $\ell' - M$. The set $\ell \cap C$ consists of exactly two points and we let Γ_1 and Γ_2 be simple loops around these points in $\ell - \ell \cap C$. If we number the Γ_i's appropriately then

$$\begin{array}{ccccc} \vartheta(\gamma) & : & \Gamma_1 & \mapsto & \Gamma_2 \\ & & \Gamma_2 & \mapsto & \Gamma_2 \Gamma_1 {\Gamma_2}^{-1} \end{array}$$

This induces the relation $\Gamma_1 = \Gamma_2$.

In case of a simple double point we may assume that C is given by the equation $z^2 = w^2$. In this case we obtain the following monodromy

$$\begin{array}{ccccc} \vartheta(\gamma) & : & \Gamma_1 & \mapsto & \Gamma_2 \Gamma_1 {\Gamma_2}^{-1} \\ & & \Gamma_2 & \mapsto & \Gamma_2 \Gamma_1 \Gamma_2 {\Gamma_1}^{-1} {\Gamma_2}^{-1} \end{array}$$

and it induces the relation $[\Gamma_1, \Gamma_2] = \Gamma_1 \Gamma_2 {\Gamma_1}^{-1} {\Gamma_2}^{-1} = 1$.

In case of a cusp we may assume that C is given by the equation $z^2 = w^3$. In this case we obtain the following monodromy

$$\begin{array}{ccccc} \vartheta(\gamma) & : & \Gamma_1 & \mapsto & \Gamma_2 \Gamma_1 \Gamma_2 {\Gamma_1}^{-1} {\Gamma_2}^{-1} \\ & & \Gamma_2 & \mapsto & \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1 {\Gamma_2}^{-1} {\Gamma_1}^{-1} {\Gamma_2}^{-1} \end{array}$$

and it induces the relation $\langle \Gamma_1, \Gamma_2 \rangle := \Gamma_1 \Gamma_2 \Gamma_1 {\Gamma_2}^{-1} {\Gamma_1}^{-1} {\Gamma_2}^{-1} = 1$.

We refer to [Ch, Partie 6.2] for more details.

A way to visualise the monodromy is as follows: We define a *half-twist* of $\ell - \ell \cap C$ to be a homeomorphism of $\ell - \ell \cap C$ with itself that is the identity outside a small disc containing the two points $\ell \cap C$ and that has the effect of turning this disc by an angle of π. In particular, a half-twist fixes the base point (z_0, w_0). Under such a half-twist Γ_1 is moved to a loop homotopic to Γ_2 and Γ_2 is moved to a loop homotopic to $\Gamma_2 \Gamma_1 {\Gamma_2}^{-1}$.

Hence the monodromy coming from a simple tangent point acts like a half-twist on Γ_1 and Γ_2. Similarly, the monodromy of a simple double point of C acts like two half-twists (a so-called *full-twist*), and the monodromy coming from a cusp corresponds to three half-twists.

This is the starting point of the *braid monodromy* introduced by Moishezon: Half-twists generate the braid group of the pair $(\ell,\, \ell - \ell \cap C)$ and so there is an induced map from $\pi_1^{\mathrm{top}}(\ell' - M)$ to this braid group describing the monodromy. Moishezon used this also for the global situation where things are getting more complicated. We refer the fearless reader to [Mo] for an introduction.

We now treat the global case of an irreducible curve C of degree d that has at most simple double points and cusps as singularities. We denote the d points of $\ell \cap C$ by $P_1, ..., P_d$ and choose in $\ell - \ell \cap C$ simple loops $\Gamma_1, ..., \Gamma_d$ around these points in $\ell - \ell \cap C$ with a common base point. As already mentioned above, these loops generate $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - C)$. The relations induced from a simple tangent point come again from a half-twist of some P_i around some P_j. This leads to a relation of the form $\Gamma_i = \gamma \Gamma_j \gamma^{-1}$ for some γ. Similarly, simple double points lead to full-twists and cusps lead to three half-twists of some P_i around some P_j.

Finally we obtain a presentation of $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - D)$ given by generators $\Gamma_1, ..., \Gamma_d$ and relations of the form

$$\begin{array}{lcll} \gamma \Gamma_i \gamma^{-1} \cdot {\Gamma_j}^{-1} & = & 1 & \text{tangent points} \\ [\gamma \Gamma_i \gamma^{-1},\, \Gamma_j] & = & 1 & \text{simple double points} \\ \langle \gamma \Gamma_i \gamma^{-1},\, \Gamma_j \rangle & = & 1 & \text{cusps.} \end{array}$$

In a given situation these γ's can be made explicit. However, for our purposes later on this is already enough. Pictures, details and quite complicated examples can be found in [Mo, Proposition 1.2] and [MoTe3, Section V].

An application of this algorithm is the case of a smooth curve C of degree d. Then $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - C)$ is generated by elements Γ_i, $i = 1, ..., d$. Then all relations come from simple tangent points as explained above and then one can show that $\Gamma_i = \Gamma_j$ for all i, j holds true. In particular, we get $\delta = {\Gamma_1}^d$ and conclude:

Proposition 6.1 *Let C be an irreducible and smooth projective curve of degree d in the projective plane. Then there are isomorphisms*

$$\pi_1^{\mathrm{top}}(\mathbb{A}^2 - C) \;\cong\; \mathbb{Z} \quad \textit{and} \quad \pi_1^{\mathrm{top}}(\mathbb{P}^2 - C) \;\cong\; \mathbb{Z}_d.$$

Severi claimed that a curve with only simple double points as singularities can be deformed “nicely“ into a union of lines in general position. Using this assertion of Severi, Zariski [Za] gave a proof of the proposition above also in the case that C is an irreducible curve that has at worst simple double points as singularities. A rigorous proof of Severi's assertion was finally given by Harris [Ha]. However, in the meantime Deligne and Fulton had come up with a proof of Zariski's conjecture that went along different lines, cf. [De].

In general, the computation of $\pi_1^{\rm top}(\mathbb{A}^2 - C)$ is a very difficult business. To get some ideas about the subtleties occurring we refer to [Dc, Section 4.4] for some classical examples.

6.2 On the fundamental group of $\mathbf{X}_{\mathbf{gal}}^{\mathbf{aff}}$

We let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n with Galois closure $f_{\rm gal} : X_{\rm gal} \to \mathbb{P}^2$. We denote by $\mathbb{A}^2$ the complement of a generic line ℓ in the projective plane $\mathbb{P}^2$. We obtain $X^{\rm aff}$ and $X_{\rm gal}^{\rm aff}$ by removing the inverse image of ℓ from X and $X_{\rm gal}$, respectively.

In Section 4.5 we constructed a short exact sequence

$$1 \;\to\; \pi_1^{\rm top}(X_{\rm gal}^{\rm aff},\, x_0) \;\to\; \pi_1^{\rm top}(X_{\rm gal}^{\rm aff},\, \mathfrak{S}_n,\, x_0) \;\to\; \mathfrak{S}_n \;\to\; 1. \qquad (*)$$

We let D be the branch curve of f and denote by $d := \deg D$ its degree. Then we choose another generic line ℓ' in $\mathbb{A}^2$. We assume that $u_0 := f_{\rm gal}(x_0)$ lies on ℓ' but not on D. The intersection of ℓ' with $\mathbb{A}^2 - D$ cuts exactly d points out of ℓ' and the inclusion $\imath$ of $\ell' - \ell' \cap D$ into $\mathbb{A}^2 - D$ induces a surjective homomorphism

$$\mathfrak{F}_d \;\cong\; \pi_1^{\rm top}(\ell' - \ell' \cap D,\, u_0) \;\overset{\imath_*}{\twoheadrightarrow}\; \pi_1^{\rm top}(\mathbb{A}^2 - D,\, u_0)$$

where $\mathfrak{F}_d$ denotes the free group of rank d. To be more precise, we can choose a good ordered system Γ_i, $i = 1, ..., d$ of generators that freely generates the group $\pi_1^{\rm top}(\ell' - \ell' \cap D, u_0)$ as explained in the previous section.

The image of Γ_i in $\pi_1^{\rm top}(\mathbb{A}^2 - D, u_0)$ is exactly a simple loop as described in Section 4.4. Since f is a good generic projection the curve D is irreducible and so the images of the Γ_i's are conjugate elements in $\pi_1^{\rm top}(\mathbb{A}^2 - D, u_0)$. As explained at the end of Section 4.4 there exists an isomorphism

$$\pi_1^{\rm top}(\mathbb{A}^2 - D,\, u_0)/ \ll \imath_*(\Gamma_i)^2 \gg \;\cong\; \pi_1^{\rm top}(X_{\rm gal}^{\rm aff},\, \mathfrak{S}_n,\, x_0). \qquad (**)$$

In Section 4.5 we defined a normal subgroup $C^{\rm aff}$ of $\pi_1^{\rm top}(X_{\rm gal}^{\rm aff},\, \mathfrak{S}_n,\, x_0)$ that was normally generated by certain commutators and triple commutators between inertia elements attached to the ramification locus $R_{\rm gal}$ of $f_{\rm gal} : X_{\rm gal}^{\rm aff} \to \mathbb{A}^2$. After taking the quotient of $(*)$ by $C^{\rm aff}$ we explained in Theorem 4.7 how to split the resulting short exact sequence.

We now combine the algorithm of Zariski and van Kampen with Theorem 4.7 and Corollary 5.11 and obtain our main result:

Theorem 6.2 *Let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n. Then there exists an isomorphism*

$$\pi_1^{\rm top}(X_{\rm gal}^{\rm aff})/C^{\rm aff} \quad \cong \quad \widetilde{\mathcal{K}}(\pi_1^{\rm top}(X^{\rm aff}),\, n).$$

If Question 2.14 has an affirmative answer for the universal cover of $X_{\rm gal}^{\rm aff}$ then $C^{\rm aff}$ is trivial.

PROOF. We keep the notations introduced so far. By abuse of notation we call $\imath_*(\Gamma_i)$ again Γ_i and we consider the following composition

$$\pi_1^{\rm top}(\ell' - \ell' \cap D) \quad \stackrel{\imath_*}{\twoheadrightarrow} \quad \pi_1^{\rm top}(\mathbb{A}^2 - D) \quad \twoheadrightarrow \quad \pi_1^{\rm top}(\mathbb{A}^2 - D)/ \ll {\Gamma_i}^2, C^{\rm aff} \gg .$$

If n denotes the degree of the good generic projection f then there is a surjective homomorphism

$$\psi \quad : \quad \pi_1^{\rm top}(\mathbb{A}^2 - D)/ \ll {\Gamma_i}^2, C^{\rm aff} \gg \quad \twoheadrightarrow \quad \mathfrak{S}_n.$$

Under the isomorphism $(**)$ the Γ_i's are identified with (conjugates) of inertia elements. In particular, ψ sends the Γ_i's to transpositions. So we can choose for each Γ_i a permutation σ_i of $\mathfrak{S}_n$ such that $\sigma_i \psi(\Gamma_i) {\sigma_i}^{-1} = (1\,2)$.

We explained in the proof of Theorem 4.7 how to find a splitting s of ψ using inertia groups. Inside $\pi_1^{\rm top}(\mathbb{A}^2 - D)/ \ll {\Gamma_i}^2, C^{\rm aff} \gg$ we define the following elements:

$$s_{i+1} \quad := \quad s(\sigma_i)\, \Gamma_i \, s(\sigma_i)^{-1} \quad \text{for } i = 1, ..., d.$$

Clearly, $s(\mathfrak{S}_n)$ and these s_i's generate the whole group.

Moreover, for every transposition $\tau_k = (k\, k+1)$ of $\mathfrak{S}_n$ there exists an element γ_k in $\pi_1^{\rm top}(\ell' - \ell' \cap D)$ such that in $\pi_1^{\rm top}(\mathbb{A}^2 - D)/ \ll {\Gamma_i}^2, C^{\rm aff} \gg$

$$s(\tau_k) \quad = \quad \imath_*(\gamma_k)\, \Gamma_1 \, \imath_*(\gamma_k)^{-1} \quad \text{for } k = 1, ..., n-1$$

holds true.

Since the s_i's are conjugate to the Γ_i's the relation ${s_i}^2 = 1$ holds true. As we have taken the quotient by $C^{\rm aff}$ also the commutator and triple commutator relations of Section 5.1 hold true. So there exists a surjective homomorphism

$$\begin{array}{ccccc} \mathcal{S}_n(d+1) & \twoheadrightarrow & \pi_1^{\rm top}(\mathbb{A}^2 - D,\, u_0)/ \ll {\Gamma_i}^2, C^{\rm aff} \gg & & \\ s_i & \mapsto & s_i & & i = 2, ..., d+1 \\ \varphi(\sigma) & \mapsto & s(\sigma) & & \sigma \in \mathfrak{S}_n \end{array}$$

where φ denotes the splitting of $\psi : \mathcal{S}_n(d) \to \mathfrak{S}_n$ we fixed in Section 5.1. In particular, we will need the fact that $\varphi((1\,2)) = s_1$.

The group $\pi_1^{\text{top}}(\ell' - \ell' \cap D)$ is freely generated by the Γ_i's. We let F be the group freely generated by elements τ_k's, $k = 1, ..., n-1$. Then we obtain surjective homomorphisms

$$\begin{array}{ccccc} \pi_1^{\text{top}}(\ell' - \ell' \cap D) * F & \stackrel{\alpha}{\twoheadrightarrow} & \mathcal{S}_n(d+1) & \stackrel{\omega}{\twoheadrightarrow} & \dfrac{\pi_1^{\text{top}}(\mathbb{A}^2 - D)}{\ll {\Gamma_i}^2, C^{\text{aff}} \gg} \\ \Gamma_i & \mapsto & \varphi(\sigma_i)^{-1} s_{i+1} \varphi(\sigma_i) & \mapsto & \Gamma_i \\ \tau_k & \mapsto & \varphi(\tau_k) & \mapsto & \gamma_k \Gamma_1 {\gamma_k}^{-1} \end{array}$$

We denote by R the kernel of $\pi_1^{\text{top}}(\ell' - \ell' \cap D)$ onto $\pi_1^{\text{top}}(\mathbb{A}^2 - D)$. Then the kernel of $\omega \circ \alpha$ is the subgroup subgroup normally generated by R, C^{aff} and the relations $\tau_k = \gamma_k \Gamma_1 {\gamma_k}^{-1}$ and ${\Gamma_i}^2$. Now since α is surjective the image of the kernel of $\omega \circ \alpha$ in $\mathcal{S}_n(d+1)$ is the kernel of ω. We note that C^{aff} and ${\Gamma_i}^2$ already lie in $\ker \alpha$.

We proved in Theorem 5.3 that for $n \geq 5$ there is an isomorphism

$$\begin{array}{cccc} \mathcal{S}_n(d+1) & \cong & \mathcal{K}(\mathfrak{F}_d,\, n) \rtimes \mathfrak{S}_n & \leq \mathfrak{F}_d{}^n \rtimes \mathfrak{S}_n \\ s_1 & \mapsto & (1\,2) & \\ s_i & \mapsto & (f_i, {f_i}^{-1}, 1, ..., 1)(1\,2) & i = 2, ..., d+1 \\ \varphi(\sigma) & \mapsto & \sigma & \forall \sigma \in \mathfrak{S}_n \end{array}$$

where $\mathfrak{F}_d$ denotes the free group of rank d, freely generated by some elements f_i, $i = 2, ..., d+1$. The goal now is to show the kernel of ω becomes an affine subgroup in the sense of Definition 5.9 under this isomorphism.

By definition of a good generic projection the branch curve D of f is irreducible and has at worst simple double points and cusps as singularities. The group $\pi_1^{\text{top}}(\mathbb{A}^2 - D, u_0)$ is generated by the Γ_i's and we have already seen in the previous section that the algorithm of Zariski and van Kampen provides us with a presentation in which all relations follow from relations of the following form:

$$\begin{array}{lcll} \gamma \Gamma_i \gamma^{-1} \cdot {\Gamma_j}^{-1} & = & 1 & \text{tangent points} \\ [\gamma \Gamma_i \gamma^{-1},\, \Gamma_j] & = & 1 & \text{simple double points} \\ \langle \gamma \Gamma_i \gamma^{-1},\, \Gamma_j \rangle & = & 1 & \text{cusps.} \end{array}$$

Under α the element Γ_i maps to $\varphi(\sigma_i)^{-1} s_{i+1} \varphi(\sigma_i)$. Thus, under α the relations coming from simple tangent points are sent to elements of the form

$$\gamma s_i \gamma^{-1} {s_j}^{-1}.$$

Such an element has to lie in the kernel of the homomorphism ψ onto $\mathfrak{S}_n$. Hence the permutation $\psi(\gamma)$ fixes $(1\,2)$. By conjugating this relation with s_j we may assume that $\psi(\gamma)$ is a permutation that is disjoint from $(1\,2)$. And after conjugating with $s(\psi(\gamma))$ we may assume that $\gamma \in \ker \psi$. Thus, we may write $\gamma = (\gamma_1, \gamma_2, ...)$ under the isomorphism of $\mathcal{S}_n(d+1)$ with $\mathcal{E}(\mathfrak{F}_d, n)$. Under this isomorphism the relation maps to

$$\gamma s_i \gamma^{-1} s_j{}^{-1} \quad \mapsto \quad (\gamma_1 f_i \gamma_2{}^{-1} f_j{}^{-1},\ \gamma_2 f_i{}^{-1} \gamma_1{}^{-1} f_j,\ 1,\ 1, ...).$$

Conjugating this element with $(1, f_j, f_j{}^{-1}, 1, ...)$ (again one of the many events where we need $n \geq 3$) we obtain an element of the form $(r, r^{-1}, 1, ...)$.

By a similar argument we see that the relations

$$\tau_k{}^{-1} \cdot \gamma_k \Gamma_1 \gamma_k{}^{-1}$$

are mapped to relations of the form $s_1 \gamma'_k s_2 \gamma'_k{}^{-1}$. As already shown above this leads to relations that are conjugate to relations of the form $(r, r^{-1}, 1, ..., 1)$.

Now we consider the relations

$$[\gamma s_i \gamma^{-1},\ s_j]$$

coming from simple double points of D. Since such a relation maps to 1 under ψ we conclude that $\psi(\gamma s_i \gamma^{-1})$ and $\gamma(s_j) = (1\,2)$ are disjoint transpositions or that $\psi(\gamma) = 1$. In the first case this relation already holds true by Lemma 5.5. In the second case we write again $\gamma = (\gamma_1, \gamma_2, ...)$ via the isomorphism of $\mathcal{S}_n(d+1)$ with $\mathcal{E}(\mathfrak{F}_d, n)$. Under this isomorphism this relation maps to

$$\begin{aligned}[\gamma s_i \gamma^{-1},\ s_j{}^{-1}] \quad \mapsto \quad & (\gamma_1 f_i \gamma_2{}^{-1} \cdot f_j{}^{-1} \cdot \gamma_1 f_i \gamma_2{}^{-1} \cdot f_j{}^{-1}, \\ & \gamma_2 f_i{}^{-1} \gamma_1{}^{-1} \cdot f_j \cdot \gamma_2 f_i{}^{-1} \gamma_1{}^{-1} \cdot f_j,\ 1,\ 1..).\end{aligned}$$

Conjugating this element with $(1, f_j, f_j{}^{-1}, 1, ...)$ we obtain an element of the form $(r, r^{-1}, 1, ...)$.

We leave it to the reader to show that also relations coming from cusps either automatically hold true or lead to relations that are conjugate to elements of the form $(r, r^{-1}, 1, ...)$.

Hence, $\pi_1^{\text{top}}(\mathbb{A}^2 - D)/ \ll \Gamma_i{}^2, C^{\text{aff}} \gg$ is the quotient of $\mathcal{S}_n(d+1)$ by a subgroup that is normally generated by elements of the form $(r, r^{-1}, 1, ..., 1)$. Thus the relations form an affine subgroup in the sense of Definition 5.9.

Corollary 5.11 tells us that the structure of this quotient is already determined by the quotient $\mathfrak{F}_d / p_1(R)$. By Theorem 4.7 and Corollary 5.13 this quotient is isomorphic to $\pi_1^{\text{top}}(X^{\text{aff}})$. Hence we conclude that there are isomorphisms

$$\begin{aligned}\pi_1^{\text{top}}(\mathbb{A}^2 - D)/ \ll \Gamma_i{}^2, C^{\text{aff}} \gg \quad & \cong \quad \pi_1^{\text{top}}(X^{\text{aff}}_{\text{gal}},\ \mathfrak{S}_n)/C^{\text{aff}} \\ & \cong \quad \widetilde{\mathcal{E}}(\pi_1^{\text{top}}(X^{\text{aff}}),\ n).\end{aligned}$$

The statement about the triviality of C^{aff} if that Question 2.14 has an affirmative answer for the universal cover of $X^{\text{aff}}_{\text{gal}}$ was already proven in Theorem 4.7. □

Corollary 6.3 *Under the assumptions of Theorem 6.2 we let $\bar{C}^{\text{aff}}$ be the image of C^{aff} in the abelianised fundamental group of $X_{\text{gal}}^{\text{aff}}$. Then there is an isomorphism*

$$H_1(X_{\text{gal}}^{\text{aff}}, \mathbb{Z})/\bar{C}^{\text{aff}} \cong H_1(X^{\text{aff}}, \mathbb{Z})^{n-1}.$$

In particular, if $\pi_1^{\text{top}}(X^{\text{aff}})$ is already abelian then we get

$$\pi_1^{\text{top}}(X_{\text{gal}}^{\text{aff}})/C^{\text{aff}} \cong \pi_1^{\text{top}}(X^{\text{aff}})^{n-1}.$$

We want to stress that these isomorphisms are not canonical.

PROOF. Corollary 5.17 tells us that the abelianisations of $\widetilde{\mathcal{K}}(-,n)$ and $\mathcal{K}(-,n)$ are isomorphic. The rest of the proof is similar to the one of Corollary 4.8 and therefore left to the reader. □

6.3 Adding the line at infinity

We have chosen a generic line ℓ in $\mathbb{P}^2$, defined $\mathbb{A}^2 := \mathbb{P}^2 - \ell$ and chosen a point $u_0 \in \mathbb{A}^2 - D$. Then we have chosen a generic line ℓ' in $\mathbb{A}^2$ containing the point u_0. We denote by $\bar{\ell}'$ the line ℓ' with point at infinity added.

The inclusion maps of topological spaces induce surjective homomorphisms of fundamental groups:

$$\begin{array}{ccc} \pi_1^{\text{top}}(\ell' - \ell' \cap D) & \twoheadrightarrow & \pi_1^{\text{top}}(\mathbb{A}^2 - D) \\ \downarrow & & \downarrow \imath_* \\ \pi_1^{\text{top}}(\bar{\ell}' - \bar{\ell}' \cap D) & \twoheadrightarrow & \pi_1^{\text{top}}(\mathbb{P}^2 - D) \end{array}$$

The group $\pi_1^{\text{top}}(\ell' - \ell' \cap D)$ is a free group of rank d freely generated by a good ordered system of generators Γ_i, $i = 1, ..., d$. By definition the element

$$\delta := \Gamma_1 \cdot ... \cdot \Gamma_d$$

is homotopic to a loop around the point at infinity of ℓ'. The subgroup normally generated by δ defines the kernel of both homomorphism downwards in the diagram above.

Proposition 6.4 *The element δ is a central element of $\pi_1^{\text{top}}(\mathbb{A}^2 - D)$ that lies in the kernel of ψ.*

Moreover, lifting δ to loops in $X_{\text{gal}}^{\text{aff}}$ and X^{aff} we obtain two short exact and central sequences

$$0 \to \langle\delta\rangle \to \pi_1^{\text{top}}(X_{\text{gal}}^{\text{aff}}) \to \pi_1^{\text{top}}(X_{\text{gal}}) \to 1$$

and

$$0 \to \langle\bar{\delta}\rangle \to \pi_1^{\text{top}}(X^{\text{aff}}) \to \pi_1^{\text{top}}(X) \to 1 \quad .$$

PROOF. For both groups $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - D)$ and $\pi_1^{\mathrm{top}}(\mathbb{P}^2 - D)$ there are surjective homomorphisms ψ onto $\mathfrak{S}_n$ that are compatible with $\imath_*$. Since δ is trivial in $\pi_1^{\mathrm{top}}(\mathbb{P}^2 - D)$ we conclude that $\psi(\delta) = 1$.

By a theorem of Oka (cf. [FL, Corollary 8.4]) the short exact sequence

$$0 \;\to\; A \;\to\; \pi_1^{\mathrm{top}}(\mathbb{A}^2 - D) \;\xrightarrow{\imath_*}\; \pi_1^{\mathrm{top}}(\mathbb{P}^2 - D) \;\to\; 1$$

is central. We know that A is normally generated by δ and hence δ must be a central element of $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - D)$. Of course δ remains central in every quotient of $\pi_1^{\mathrm{top}}(\mathbb{A}^2 - D)$.

We recall the short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) & \to & \pi_1^{\mathrm{top}}(\mathbb{A}^2 - D)/\ll {\Gamma_i}^2 \gg & \xrightarrow{\psi} & \mathfrak{S}_n & \to & 1 \\ & & \downarrow & & \downarrow \imath_* & & || & & \\ 1 & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}) & \to & \pi_1^{\mathrm{top}}(\mathbb{P}^2 - D)/\ll {\Gamma_i}^2 \gg & \xrightarrow{\psi} & \mathfrak{S}_n & \to & 1 \end{array}$$

We already noted that the kernel of the surjective homomorphism $\imath_*$ is generated by δ. Since $\psi(\delta) = 1$ the loop δ lies in $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})$. This yields the first exact sequence.

There exist surjective homomorphisms

$$\begin{array}{ccc} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}, \mathfrak{S}_{n-1}^{(i)}) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(X^{\mathrm{aff}}) \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}, \mathfrak{S}_{n-1}^{(i)}) & \twoheadrightarrow & \pi_1^{\mathrm{top}}(X) \end{array}$$

The kernel of the upper horizontal homomorphism N is generated by inertia groups. The kernel of the lower horizontal homomorphism is generated by the image of N from above. The kernel of the left arrow downwards is generated by δ. Chasing around this diagram we find that the kernel of the surjective map from $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})$ onto $\pi_1^{\mathrm{top}}(X)$ is generated by $\bar{\delta}$. □

In Theorem 6.2 we constructed an isomorphism

$$\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} \;\cong\; \widetilde{\mathcal{K}}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}),\, n).$$

Since δ is central it is stable under the $\mathfrak{S}_n$-action on the right. The same holds true when passing to the quotient $\mathcal{K}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}), n)$. So if we consider $\mathcal{K}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}), n)$ as a subgroup of $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})^n$ then δ maps to an element of the diagonal. And Proposition 6.4 tells us exactly what this element is:

Proposition 6.5 *Under the isomorphism of Theorem 6.2 and the surjective map of Theorem 4.7 the loop δ maps as follows*

$$\begin{array}{ccccc} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & \widetilde{\mathcal{K}}(\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}),\, n) & \twoheadrightarrow & \mathcal{K}(\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}), n) \\ \delta & \mapsto & \delta & \mapsto & (\bar{\delta}, ..., \bar{\delta}) \end{array}$$

where $\bar{\delta}$ is the central element of Proposition 6.4. □

Again, we can say a little bit more for the abelianisation

Corollary 6.6 *We keep the notations and assumptions of Theorem 6.2. Then we denote by $\bar{C}^{\mathrm{proj}}$ the image of C^{proj} in the abelianised fundamental group of X_{gal}. Then there exists an isomorphism*

$$H_1(X_{\mathrm{gal}},\, \mathbb{Z})/\bar{C}^{\mathrm{proj}} \;\cong\; H_1(X,\, \mathbb{Z}) \,\oplus\, H_1(X^{\mathrm{aff}},\, \mathbb{Z})^{n-2}.$$

In particular, if $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})$ is abelian then

$$\pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} \;\cong\; \pi_1^{\mathrm{top}}(X) \,\times\, \pi_1^{\mathrm{top}}(X^{\mathrm{aff}})^{n-2}.$$

We note that these isomorphisms are not canonical.

PROOF. To increase readability, we abbreviate $H_1(-,\, \mathbb{Z})$ just by $H_1(-)$.

Since abelianisation is not an exact functor we have to proceed by hand:

$$\begin{array}{ccccccccc} 1 & \to & \langle\delta\rangle & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) & \to & \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \langle\delta'\rangle & \hookrightarrow & H_1(X_{\mathrm{gal}}^{\mathrm{aff}}) & \twoheadrightarrow & H_1(X_{\mathrm{gal}}) & & \end{array}$$

where δ' denotes the image of δ in $H_1(X_{\mathrm{gal}}^{\mathrm{aff}})$. Let x be an element of $H_1(X_{\mathrm{gal}}^{\mathrm{aff}})$ that maps to 0 in $H_1(X_{\mathrm{gal}})$. We can lift this to an element $\tilde{x}$ of $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})$ that has to map to a product of commutators in $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}})$ by commutativity of the diagram. But this means that $\tilde{x}$ is a product of δ^s for some integer s times some commutators. Changing $\tilde{x}$ by commutators we still get a lift of x. So we may assume that $\tilde{x}$ actually equals δ^s. Therefore, x is equal to $\bar{\delta}^s$. This shows that we have an exact sequence

$$1 \;\to\; \langle\bar{\delta}\rangle \;\to\; H_1(X_{\mathrm{gal}}^{\mathrm{aff}}) \;\to\; H_1(X_{\mathrm{gal}}) \;\to\; 1.$$

We denote by $\bar{\delta}$ the image of δ in $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})$ we know from Proposition 6.4 that the subgroup generated by $\bar{\delta}$ inside $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})$ is equal to the kernel of the projection $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}}) \twoheadrightarrow \pi_1^{\mathrm{top}}(X)$. So we obtain another exact sequence

$$1 \;\to\; \langle\bar{\delta}'\rangle \;\to\; H_1(X^{\mathrm{aff}}) \;\to\; H_1(X) \;\to\; 1$$

where $\bar{\delta}'$ denotes the image of δ' in $H_1(X^{\text{aff}})$.

Using Theorem 6.2 we know that there is an embedding

$$H_1(X^{\text{aff}}_{\text{gal}})/\bar{C}^{\text{aff}} \quad \hookrightarrow \quad H_1(X^{\text{aff}})^n$$

that sends δ' to $(\bar{\delta}', ..., \bar{\delta}')$ by Proposition 6.5. The image of this homomorphism equals the subgroup given by (written multiplicatively)

$$\{(x_1, ..., x_n) \mid \prod_{i=1}^{n} x_i = 1\}$$

and is abstractly isomorphic to $H_1(X^{\text{aff}})^{n-1}$ by Corollary 3.5. This isomorphism is given by projecting onto the last $n-1$ factors of $H_1(X^{\text{aff}})^n$. The element δ' maps to $(\bar{\delta}', ..., \bar{\delta}')$ ($n-1$ factors) under this projection.

So we conclude that $H_1(X_{\text{gal}})$ is isomorphic to the quotient of $H_1(X^{\text{aff}})^{n-1}$ by the subgroup generated by $(\bar{\delta}', ..., \bar{\delta}')$ in it. Since the quotient $H_1(X^{\text{aff}})$ by the subgroup generated by $\bar{\delta}'$ is isomorphic to $H_1(X)$ the result follows from the following lemma. □

Lemma 6.7 *Let G be an abelian group and N be a subgroup. Let $n \geq 2$ be a natural number. We let $\Delta : G \to G^n$ be the diagonal embedding of G into G^n given by $g \mapsto (g, ..., g)$. Then there exists a non-canonical isomorphism*

$$G^n/\Delta(N) \quad \cong \quad G^{n-1} \times G/N.$$

PROOF. We define the map (written multiplicatively)

$$\begin{array}{cccc} \psi_1 & : & G^n & \to & G^{n-1} \\ & & (g_1, ..., g_n) & \mapsto & (g_2 g_1^{-1}, ..., g_n g_1^{-1}). \end{array}$$

Since G is abelian this defines a homomorphism of groups. The kernel of ψ_1 is equal to the diagonal embedding $\Delta(G)$ of G inside G^n.

We denote by $\psi_2 : G \to G/N$ be the natural quotient map. Then the kernel of the homomorphism

$$\begin{array}{cccccc} \psi & : & G^n & \to & G^{n-1} & \times & G/N \\ & & \vec{g} = (g_1, ..., g_n) & \mapsto & (\psi_1(\vec{g}) & , & \psi_2(g_1)) \end{array}$$

is equal to $\Delta(G) \cap (N \times G^{n-1}) = \Delta(N)$. We leave it to the reader to show surjectivity.

We finally note that we have somehow “favoured“ the first component when we constructed this isomorphism and this is what destroyed the symmetry. □

Remark 6.8 *Proposition 6.5 shows us that $\pi_1^{\rm top}(X_{\rm gal})/C^{\rm proj}$ is the quotient of $\widetilde{\mathcal{K}}(\pi_1^{\rm top}(X^{\rm aff}), n)$ by the cyclic group $\Delta(\langle\delta\rangle)$.*

From this we see that the natural surjection

$$\begin{array}{rcl} \pi_1^{\rm top}(X_{\rm gal})/C^{\rm proj} & \cong & \widetilde{\mathcal{K}}(\pi_1^{\rm top}(X^{\rm aff}), n)/\Delta(\langle\delta\rangle) \\ & \twoheadrightarrow & \widetilde{\mathcal{K}}(\pi_1^{\rm top}(X^{\rm aff})/\langle\delta'\rangle, n) \qquad \cong \quad \widetilde{\mathcal{K}}(\pi_1^{\rm top}(X), n) \end{array}$$

given by Theorem 4.7 need not be an isomorphism for projective surfaces. In fact, Proposition 7.1 gives us an example where this is not the case.

6.4 Generic projections from simply connected surfaces

We start with a result that should be well-known but the author could not find a reference for it.

Proposition 6.9 *Let X be smooth projective surface that is simply connected. Let D be an smooth and ample divisor on X.*

Let d be the maximum

$$d(\mathcal{L}) \;:=\; \max\{m \,|\, \exists \mathcal{M}, \mathcal{M}^{\otimes m} \cong \mathcal{L}\}$$

Then $d(\mathcal{L})$ exists (i.e. there is a finite maximum) and there is an isomorphism

$$\pi_1^{\rm top}(X - D) \;\cong\; \mathbb{Z}_{d(\mathcal{L})}.$$

In particular, this group is always a finite cyclic group.

PROOF. By a theorem of Nori [N, Corollary 2.5] we know that $\pi_1^{\rm top}(X - D)$ is abelian.

But since D is smooth and irreducible (ample implies connected and being connected and smooth implies irreducible) every cover branched along D is a cyclic Galois cover. Such a cover of order c is given by a line bundle $\mathcal{F}$ and an isomorphism $\mathcal{F}^{\otimes c} \cong \mathcal{O}_X(D)$.

On the other hand, $\pi_1^{\rm top}(X - D)$ is a finitely generated group and with maximal finite quotient $\mathbb{Z}_d$ where $d = d(\mathcal{L})$ as defined above.

To see that d is actually a well-defined and finite number we consider the long exact cohomology sequence associated to the exponential sequence:

$$\ldots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \underbrace{H^1(X, \mathcal{O}_X^*)}_{\cong \mathrm{Pic}(X)} \stackrel{c_1}{\to} H^2(X, \mathbb{Z}) \to \ldots$$

Since X is simply connected its first Betti number vanishes and so we conclude from Hodge theory that $H^1(X, \mathcal{O}_X) = 0$. Hence the map $c_1 : \mathrm{Pic}(X) \to$

$H^2(X, \mathbb{Z})$ is an injection. The latter group is a finitely generated abelian group and so $\mathrm{Pic}(X)$ must be.

Actually, $\mathrm{Pic}(X)$ is a free abelian group: By the universal coefficient formula in algebraic topology all torsion in $H^2(X, \mathbb{Z})$ comes from the torsion of $H_1(X, \mathbb{Z})$ which is the abelianised fundamental group of X. Since we assumed that X is simply connected there is no torsion in $H^2(X, \mathbb{Z})$ and so also $\mathrm{Pic}(X)$ is without torsion being a subgroup of $H^2(X, \mathbb{Z})$.

From this it follows that d is a well-defined finite number. □

Definition 6.10 *We call the number d associated to an ample divisor D on a simply connected surface the* **divisibility index** *of D.*

As an easy consequence we get the following

Theorem 6.11 *Assume that $f : X \to \mathbb{P}^2$ is a good generic projection of degree n given by a sufficiently ample line bundle $\mathcal{L}$. Assume furthermore that X is simply connected.*

We denote by $d := d(\mathcal{L})$ the divisibility index of $\mathcal{L}$. Keeping the notations of Theorem 6.2 there are isomorphisms

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & \mathbb{Z}_d{}^{n-1} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \cong & \mathbb{Z}_d{}^{n-2}. \end{array}$$

In particular, these quotients are both finite and abelian.

PROOF. For a generic projection $f : X \to \mathbb{P}^2$ the inverse image of a generic line ℓ on X is a smooth and ample curve by Bertini's theorem. To be more precise, we have $\mathcal{O}_X(f^{-1}(\ell)) \cong \mathcal{L}$.

Applying Corollary 6.3 and Corollary 6.6 to Proposition 6.9 we get the result. □

Remark 6.12 *This result is similar to the one obtained in [ADKY]. However, there they consider a different quotient than we do and use the technique of braid monodromy factorisations in the setup of symplectic topology.*

6.5 A purely topological description of the Galois closure

Given a good generic projection $f : X \to \mathbb{P}^2$ of degree n with Galois closure X_{gal} there is an action of $\mathfrak{S}_n$ on X_{gal}. We denote by D the branch locus of f and note that its ramification index with respect to $f_{\mathrm{gal}} : X_{\mathrm{gal}} \to \mathbb{P}^2$ equals 2. We have seen in Section 4.4 that there exists an isomorphism

$$\pi_1^{\mathrm{orb}}(\mathbb{P}^2, \{D, 2\}, f_{\mathrm{gal}}(x_0)) \;\cong\; \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}, \mathfrak{S}_n, x_0).$$

We remark that the group on the left is the starting point for the computation of $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}})$ in [MoTe1].

We define $u_0 := f_{\mathrm{gal}}(x_0)$. Every loop inside $\mathbb{P}^2 - D$ based at u_0 can be lifted to the n points of $f^{-1}(u_0)$. The resulting paths inside $X - f^{-1}(D)$ yield a permutation of the set $f^{-1}(u_0)$. This defines a homomorphism

$$\psi \quad : \quad \pi_1^{\mathrm{top}}(\mathbb{P}^2 - D,\, u_0) \quad \to \quad \mathfrak{S}_n.$$

This is of course the same homomorphism as the one constructed in Section 4.4 and so it is surjective.

In [Mi] and [MoTe1] the Galois closure of a generic projection was defined in a slightly different way. From [SGA1, Exposé V.4.g] it follows that their definition defines the same object. For the sake of completeness we decided to include a proof in the topological setup:

Proposition 6.13 *Let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n.*

1. *As a topological space $X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)$ is homeomorphic to the subspace*

$$\{(a_1, ..., a_n) \,|\, a_i \neq a_j,\, f(a_i) = f(a_j) \,\forall i \neq j\} \quad \subset \quad (X - f^{-1}(D))^n.$$

2. *As a topological space X_{gal} is homeomorphic to the closure of $X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)$ inside X^n.*

Moreover, there are unique analytic structures on these spaces that are in fact algebraic making the homeomorphisms above algebraic isomorphisms.

PROOF. The map from $X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)$ to $\mathbb{P}^2 - D$ is a regular topological cover with group $\mathfrak{S}_n$. Hence there is a short exact sequence

$$1 \quad \to \quad \pi_1^{\mathrm{top}}(X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)) \quad \to \quad \pi_1^{\mathrm{top}}(\mathbb{P}^2 - D) \quad \stackrel{\varphi}{\to} \quad \mathfrak{S}_n \quad \to \quad 1.$$

Moreover, $X - f^{-1}(D)$ is a cover lying in between. Also, $X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)$ is the Galois closure of the topological cover $X - f^{-1}(D) \to \mathbb{P}^2 - D$ since the function fields are the same as for the projective surfaces. This now coincides with the topological notion of a regular cover associated to a given cover. Hence up to conjugation ψ and φ are equal.

We let $Z \subset (X - f^{-1}(D))^n$ be the space defined in the second assertion. There is a fixed point free action of $\mathfrak{S}_n$ on Z with quotient $\mathbb{P}^2 - D$. This is exactly the principal fibre bundle with fibre $\mathfrak{S}_n$ associated to the homomorphism ψ. Since ψ is surjective Z is connected. So there must be a homeomorphism between Z and $X_{\mathrm{gal}} - {f_{\mathrm{gal}}}^{-1}(D)$ since we can identify their fundamental groups with the same subgroup of $\pi_1^{\mathrm{top}}(\mathbb{P}^2 - D)$ and the inclusions are induced from the respective cover maps.

We look at the projection $f^n : X^n \to (\mathbb{P}^2)^n$ and consider the diagonal embedding Δ of D in $(\mathbb{P}^2)^n$. To obtain the closure of Z inside X^n we have to glue in points above Δ.

In the proof of [Fa, Proposition 1], Faltings computed how $X \times_{\mathbb{P}^2} X \to \mathbb{P}^2$ locally looks like above points of D. The same local computations applied to the n-fold fibre product $X \times_{\mathbb{P}^2} ... \times_{\mathbb{P}^2} X$ show that we can complete Z to a smooth analytic surface $\bar{Z}$. The complement $\bar{Z} - Z$ is a divisor on $\bar{Z}$ that locally looks the same like R_{gal} in X_{gal}. So there is only one way to define a map of topological spaces from $\bar{Z}$ to X_{gal} compatible with the projections to $\mathbb{P}^2 - D$ and the respective embeddings of $X_{\text{gal}} - {f_{\text{gal}}}^{-1}(D)$. Also the local analytic structure can be made compatible giving a map of analytic spaces $\bar{Z} \to X_{\text{gal}}$. Since $\bar{Z}$ is a compact subspace of the projective space X^n this map is projective and hence algebraic by [GAGA]. □

Composing the homomorphism from $\pi_1^{\text{top}}(\mathbb{A}^2 - D, u_0)$ onto $\pi_1^{\text{top}}(\mathbb{P}^2 - D, u_0)$ with the homomorphism ψ from the latter group onto $\mathfrak{S}_n$ we obtain a homomorphism that we will also call ψ:

$$\psi \quad : \quad \pi_1^{\text{top}}(\mathbb{A}^2 - D,\, u_0) \quad \to \quad \mathfrak{S}_n.$$

Clearly, it is also surjective. Furthermore we can identify $X_{\text{gal}}^{\text{aff}} - {f_{\text{gal}}}^{-1}(D)$ with the space

$$\{(a_1, ..., a_n) \,|\, a_i \neq a_j,\ f(a_i) = f(a_j)\, \forall i \neq j\} \quad \subset \quad (X^{\text{aff}} - f^{-1}(D))^n.$$

A remark on symmetric products

For a natural number $n \geq 2$ and a topological space Z there is an action of the symmetric group $\mathfrak{S}_n$ on Z^n given by permuting the factors. By definition the *n.th symmetric product* of Z is the quotient

$$\text{Sym}^n(Z) \quad := \quad \mathfrak{S}_n \backslash Z^n.$$

If we choose a point $(z, ..., z)$ on the diagonal inside Z^n its inertia group is the whole symmetric group. Using the inertia group at this point we obtain a splitting of the short exact sequence

$$1 \quad \to \quad \pi_1^{\text{top}}(Z^n) \quad \to \quad \pi_1^{\text{top}}(Z^n, \mathfrak{S}_n) \quad \to \quad \mathfrak{S}_n \quad \to \quad 1$$

Under an appropriate isomorphism of $\pi_1^{\text{top}}(Z^n)$ with $\pi_1^{\text{top}}(Z)^n$ the action of $\mathfrak{S}_n$ on $\pi_1^{\text{top}}(Z^n)$ is given by permutation of the factors of $\pi_1^{\text{top}}(Z)^n$.

So we are in the situation considered in Section 3.1. Hence the kernel of the homomorphism from $\pi_1^{\text{top}}(Z^n, \mathfrak{S}_n)$ onto $\pi_1^{\text{top}}(\text{Sym}^n(Z))$ can be identified with

$\mathcal{E}(\pi_1^{\mathrm{top}}(Z), n)$. By Corollary 3.3 we there exists a surjective homomorphism and an isomorphism

$$\pi_1^{\mathrm{top}}(Z)^n \twoheadrightarrow \pi_1^{\mathrm{top}}(Z)^{\mathrm{ab}} \cong \pi_1^{\mathrm{top}}(\mathrm{Sym}^n(Z)).$$

This was (in a slightly different form) already remarked in [SGA1, Remarque IX.5.8].

So let $f : X \to \mathbb{P}^2$ be a good generic projection of degree n. We have seen in Proposition 6.13 that the Galois closure X_{gal} occurs as a subspace of X^n. Also the action of $\mathfrak{S}_n$ on X^n coincides with the one on X_{gal}. Taking the quotient by $\mathfrak{S}_n$ we obtain maps

$$\begin{array}{ccc} X^n & \twoheadrightarrow & \mathrm{Sym}^n(X) \\ \uparrow \imath & & \uparrow \\ X_{\mathrm{gal}} & \twoheadrightarrow & \mathbb{P}^2 \end{array}$$

where the maps upwards are inclusion maps of topological spaces. It is known that the n.th symmetric product of a smooth algebraic surface has singularities as soon as $n \geq 2$.

From the commutativity of this diagram we conclude that

$$\begin{array}{rcl} \imath_* \left(\pi_1^{\mathrm{top}}(\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}))\right) & \subseteq & \ker\left(\pi_1^{\mathrm{top}}(X^n) \to \pi_1^{\mathrm{top}}(\mathrm{Sym}^n(X))\right) \\ & \cong & \mathcal{K}(\pi_1^{\mathrm{top}}(X),\, n). \end{array}$$

In this setup Theorem 4.7 says that the homomorphism $\imath_*$ is surjective. Clearly, everything also works in the affine situation.

It is tempting to think of $(X^{\mathrm{aff}})^n \to \mathrm{Sym}^n(X^{\mathrm{aff}})$ as something that is close to an algebraic fibre bundle with typical fibre $X_{\mathrm{gal}}^{\mathrm{aff}}$. Then it would be natural to expect an exact sequence of homotopy groups

$$\ldots \to \underbrace{\pi_2^{\mathrm{top}}(\mathbb{A}^2)}_{=\{1\}} \stackrel{?}{\to} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}}) \to \pi_1^{\mathrm{top}}(X^{\mathrm{aff}})^n \to \underbrace{\pi_1^{\mathrm{top}}(\mathrm{Sym}^n(X))}_{\cong \pi_1^{\mathrm{top}}(X)^{\mathrm{ab}}} \to 1.$$

However, Theorem 6.2 tells us that in the affine case $\pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})$ is in general not a subgroup of $\pi_1^{\mathrm{top}}(X^{\mathrm{aff}})^n$ even though the rest of this sequence is exact.

The author does not know whether this point of view may nevertheless shed new light on the whole problem of determining the fundamental groups of X_{gal} and $X_{\mathrm{gal}}^{\mathrm{aff}}$.

7 Examples

7.1 $\mathbb{P}^2$

Let $X := \mathbb{P}^2$ be the complex projective plane.

For $k \geq 5$ the line bundle $\mathcal{L}_k := \mathcal{O}_{\mathbb{P}^2}(k)$ is sufficiently ample by Lemma 2.2. Combining Proposition 2.5 with Proposition 2.8 we see that a generic three-dimensional linear subspace of $H^0(\mathbb{P}^2, \mathcal{L}_k)$ gives rise to a good generic projection that we denote by f_k.

Proposition 7.1 *Let X_{gal} be the Galois closure of a good generic projection f_k. Then there are isomorphisms*

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & {\mathbb{Z}_k}^{k^2-1} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \cong & {\mathbb{Z}_k}^{k^2-2}. \end{array}$$

PROOF. The morphism f_k has degree $n = \deg f_k = k^2$. The divisibility index of $\mathcal{L}_k$ in $\mathrm{Pic}(X)$ is k and we only have to plug in this data into Theorem 6.11. □

Remark 7.2 *The results of Moishezon and Teicher [MoTe2] show that C^{aff} and C^{proj} are trivial.*

7.2 $\mathbb{P}^1 \times \mathbb{P}^1$

Let $X := \mathbb{P}^1 \times \mathbb{P}^1$.

For $a \geq 5$ and $b \geq 5$ the line bundle $\mathcal{L}_{(a,b)} := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$ is sufficiently ample, cf. Lemma 2.2. Combining Proposition 2.5 with Proposition 2.8 we see that a generic three-dimensional linear subspace of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L}_{(a,b)})$ gives rise to a good generic projection that we denote by $f_{(a,b)}$.

Proposition 7.3 *Let X_{gal} be the Galois closure of a good generic projection $f_{(a,b)}$. Then there are isomorphisms*

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & {\mathbb{Z}_{\gcd(a,b)}}^{2ab-1} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \cong & {\mathbb{Z}_{\gcd(a,b)}}^{2ab-2}. \end{array}$$

PROOF. The morphism $f_{(a,b)}$ has degree $n = \deg f_{(a,b)} = 2ab$. The divisibility index of $\mathcal{L}_{(a,b)}$ in $\mathrm{Pic}(X)$ is $\gcd(a,b)$ and we only have to plug in this data into Theorem 6.11. □

Remark 7.4 *The results of Moishezon and Teicher [MoTe1] and [MoTe4] show that C^{aff} and C^{proj} are trivial.*

7.3 Surfaces in $\mathbb{P}^3$

Let X_m be a smooth surface of degree $m \geq 2$ in $\mathbb{P}^3$.

For $k \geq 5$ the line bundle $\mathcal{L}_k := \mathcal{O}_{\mathbb{P}^3}(k)|_{X_m}$ is sufficiently ample, cf. Lemma 2.2. Combining Proposition 2.5 with Proposition 2.8 we see that a generic three-dimensional linear subspace of $H^0(X_m, \mathcal{L}_k)$ gives rise to a good generic projection that we denote by $f_k : X_m \to \mathbb{P}^2$.

Proposition 7.5 *Let X_{gal} be the Galois closure of a good generic projection f_k. Then there are isomorphisms*

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & \mathbb{Z}_k{}^{mk^2-1} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \cong & \mathbb{Z}_k{}^{mk^2-2}. \end{array}$$

PROOF. The morphism f_k has degree $n = \deg f_k = mk^2$.

Lefschetz's theorem on hyperplane sections tells us that the surface X_m is simply connected. We let C be a smooth section of $\mathcal{O}_{\mathbb{P}^3}(1)|_{X_m}$. The surface $X_m - C$ is simply connected by [N, Example 6.8]. So the divisibility index of $\mathcal{L}_1$ equals 1 for and hence this index is equal to k for $\mathcal{L}_k$.

Applying Theorem 6.11 we get the result. □

7.4 Hirzebruch surfaces

Let $X := \mathbb{F}_e := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with $e \geq 2$ be the e.th Hirzebruch surface.

We denote by F the class of a fibre of $X \to \mathbb{P}^1$ and by H the class of the tautological bundle $\mathcal{O}_{\mathbb{F}_e}(1)$ in $\mathrm{Pic}(\mathbb{F}_e)$. We refer to [Hart, Section V.2] for details on the intersection theory and the canonical line bundle of Hirzebruch surfaces.

For $a > 0$ and $b > ae$ the line bundle $\mathcal{L}_{(a,b)} := \mathcal{O}_{\mathbb{F}_e}(aH + bF)$ on $\mathbb{F}_e$ is ample by [Hart, Theorem V.2.17]. We assume that $\mathcal{L}_{(a,b)}$ is sufficiently ample which can be achieved by taking a tensor product of at least five very ample line bundles cf. Lemma 2.2. If Proposition 2.8 assures the existence of simple double points then we denote by $f_{(a,b)} : \mathbb{F}_e \to \mathbb{P}^2$ the good generic projection associated to a generic three-dimensional linear subspace of $H^0(\mathbb{F}_e, \mathcal{L}_{(a,b)})$.

Proposition 7.6 *Assume that $f_{(a,b)} : \mathbb{F}_e \to \mathbb{P}^2$ is a good generic projection. We let X_{gal} be the Galois closure of $f_{(a,b)}$. Then there are isomorphisms*

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})/C^{\mathrm{aff}} & \cong & \mathbb{Z}_{\gcd(a,b)}{}^{2ab+ea^2-1} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})/C^{\mathrm{proj}} & \cong & \mathbb{Z}_{\gcd(a,b)}{}^{2ab+ea^2-2}. \end{array}$$

PROOF. The morphism $f_{(a,b)}$ has degree $n = \deg f_{(a,b)} = 2ab + ea^2$. The divisibility index of $\mathcal{L}_{(a,b)}$ in $\mathrm{Pic}(X)$ is $\gcd(a,b)$ and we only have to plug in this data into Theorem 6.11. □

Remark 7.7 *Using the results of Moishezon, Teicher and Robb [MoTeRo] we see that C^{aff} and C^{proj} are trivial.*

7.5 Geometrically ruled surfaces

We let C be a smooth projective curve of genus g and we let $\mathcal{E}$ be a rank 2 vector bundle on C. We assume that $H^0(C, \mathcal{E}) \neq 0$ but that for all line bundles $\mathcal{L}$ with negative degree the bundle $\mathcal{E} \otimes \mathcal{L}$ has no non-trivial global sections.

Then we define $\pi : X := \mathbb{P}(\mathcal{E}) \to C$ to be the projectivisation of $\mathcal{E}$ and $e := -\deg \mathcal{E}$. This is a geometrically ruled surface over C with invariant e. Conversely, by [Hart, Proposition V.2.8] every geometrically ruled surface over a curve is the projectivisation of a rank 2 vector bundle that fulfills the above assumptions on the global sections.

The Picard group of X is isomorphic to $\mathbb{Z} \oplus \mathrm{Pic}(C)$. It is generated by the pull-back of $\mathrm{Pic}(C)$ and by the class C_0 of a section of π with $\mathcal{O}_X(C_0)$ isomorphic to the tautological line bundle $\mathcal{O}_X(1)$ on X. We choose a natural number $k > 0$ and a line bundle $\mathcal{L}_C$ on C of degree $\deg \mathcal{L}_C > ke$. Then we define the line bundle $\mathcal{L}_X$ on X to be

$$\mathcal{L}_X \quad := \quad \mathcal{O}_X(C_0)^{\otimes k} \otimes \pi^*(\mathcal{L}_C).$$

This line bundle is ample by [Hart, Proposition V.2.20] and [Hart, Proposition V.2.21]. We assume that $\mathcal{L}_X$ is sufficiently ample which can be achieved by taking the tensor product of at least five very ample line bundles, cf. Lemma 2.2. If Proposition 2.8 assures the existence of simple double points then we denote by $f_{\mathcal{L}_X} : X \to \mathbb{P}^2$ the good generic projection associated to a generic three-dimensional linear subspace of $H^0(X, \mathcal{L}_X)$. The degree of $f_{\mathcal{L}_X}$ equals the self-intersection of $\mathcal{L}_X$

$$n \quad := \quad \deg f_{\mathcal{L}_X} \quad = \quad 2k \deg \mathcal{L}_C - ek^2.$$

Also we denote by

$$d(\mathcal{L}_X) \quad := \quad \max\{m \in \mathbb{Z} \,|\, \exists \mathcal{M} \in \mathrm{Pic}(X),\, \mathcal{M}^{\otimes m} \cong \mathcal{L}_X\}$$

the divisibility index of $\mathcal{L}_X$ in $\mathrm{Pic}(X)$. This number divides the greatest common divisor $\gcd\{k, \deg \mathcal{L}_C\}$.

Proposition 7.8 *Let X be a geometrically ruled surface over a curve of genus g and let $\mathcal{L}_X$ be the line bundle considered above.*

We assume that $\mathcal{L}_X$ is sufficiently ample and that $f_{\mathcal{L}_X} : X \to \mathbb{P}^2$ is a good generic projection. We let X_{gal} be the Galois closure of $f_{\mathcal{L}_X}$. Then there are isomorphisms

$$\begin{array}{rcl} \pi_1^{\mathrm{top}}(X_{\mathrm{gal}}^{\mathrm{aff}})^{\mathrm{ab}}/\bar{C}^{\mathrm{aff}} & \cong & \mathbb{Z}_{d(\mathcal{L}_X)}{}^{n-1} \oplus \mathbb{Z}^{2g(n-1)} \\ \pi_1^{\mathrm{top}}(X_{\mathrm{gal}})^{\mathrm{ab}}/\bar{C}^{\mathrm{proj}} & \cong & \mathbb{Z}_{d(\mathcal{L}_X)}{}^{n-2} \oplus \mathbb{Z}^{2g(n-1)}. \end{array}$$

PROOF. If we denote by Π_g the fundamental group of C (cf. Section 1.1) then $\pi_1^{\text{top}}(X)$ is isomorphic to Π_g since X is a smooth surface that is birational to the product $\mathbb{P}^1 \times C$.

Proposition 6.4 tells us that there is a central short exact sequence

$$0 \;\to\; Z \;\to\; \pi_1^{\text{top}}(X^{\text{aff}}) \;\to\; \underbrace{\pi_1^{\text{top}}(X)}_{\cong \Pi_g} \;\to\; 1$$

where Z is a cyclic group. If we abelianise we obtain a short exact sequence

$$0 \;\to\; \phi(Z) \;\to\; \pi_1^{\text{top}}(X^{\text{aff}})^{\text{ab}} \;\to\; \underbrace{\pi_1^{\text{top}}(X)^{\text{ab}}}_{\cong \mathbb{Z}^{2g}} \;\to\; 0$$

where ϕ denotes the homomorphism from $\pi_1^{\text{top}}(X^{\text{aff}})$ onto its abelianisation. Since the quotient group on the right is a free abelian group we can split this short exact sequence and obtain a non-canonical isomorphism

$$\pi_1^{\text{top}}(X^{\text{aff}})^{\text{ab}} \;\cong\; \phi(Z) \oplus \mathbb{Z}^{2g}.$$

Then the direct summand $\phi(Z)$ occurs as a quotient of $\pi_1^{\text{top}}(X)^{\text{ab}}$. This quotient describes cyclic covers branched along $H := {f_{\mathcal{L}_X}}^{-1}(\ell)$ where ℓ is a generic line in $\mathbb{P}^2$. This H is a smooth and irreducible divisor and so to give a cyclic cover branched along H is the same as to give a line bundle $\mathcal{M}$ and an isomorphism $\mathcal{M}^{\otimes m} \cong \mathcal{O}_X(H)$. Since $\mathcal{O}_X(H)$ is isomorphic to $\mathcal{L}_X$ we see that the maximal cyclic cover possible is of degree $d(\mathcal{L}_X)$. Hence

$$\phi(Z) \;\cong\; \mathbb{Z}_{d(\mathcal{L}_X)}$$

and so we found the structure of $\pi_1^{\text{top}}(X^{\text{aff}})^{\text{ab}}$.

Using Corollary 6.3 we obtain an isomorphism

$$\pi_1^{\text{top}}(X_{\text{gal}}^{\text{aff}})^{\text{ab}}/\bar{C}^{\text{aff}} \;\cong\; {\mathbb{Z}_{d(\mathcal{L}_X)}}^{n-1} \oplus \mathbb{Z}^{2g(n-1)}$$

and using Corollary 6.6 we obtain the structure of the abelianised quotient in the projective setup. □

7.6 An instructive counter-example

We consider again the projective plane $\mathbb{P}^2$ but this time together with the line bundle $\mathcal{L}_2 := \mathcal{O}_{\mathbb{P}^2}(2)$. The image of $\mathbb{P}^2$ in $\mathbb{P}^5$ with respect to $\mathcal{L}_2$ is usually called the *second Veronese surface*.

We denote by $f_2 : \mathbb{P}^2 \to \mathbb{P}^2$ a sufficiently general projection from this Veronese surface onto a linearly embedded $\mathbb{P}^2$ inside $\mathbb{P}^5$.

Moishezon and Teicher [MoTe2, Proposition 2] computed the fundamental group of the Galois closure $X_{\rm gal}$ of f_2:

$$\pi_1^{\rm top}(X_{\rm gal}) \;\cong\; \mathbb{Z}^2.$$

The interesting point is that the branch curve of f_2 has 9 cusps but no simple double points. But this means that if τ_1 and τ_2 are two disjoint transpositions then the curves R_{τ_1} and R_{τ_2} do not intersect. So Question 2.14 has a negative answer already for the trivial cover $X_{\rm gal} \to X_{\rm gal}$. And indeed the quotient computed by our method is ${\mathbb{Z}_2}^2$.

This is in fact the only example known to the author where $C^{\rm aff}$ is non-trivial and the quotient computed by Theorem 6.2 is not isomorphic to the fundamental group of the Galois closure.

This example suggests that the existence of simple double points on the branch curves really is essential.

However, the second Veronese surface arises in many situations as a counter-example and there are several classical theorems in classical algebraic geometry that have to exclude this surface to be true. For example, this surface and its projection onto $\mathbb{P}^2$ would also be a counter-example to Chisini's conjecture (Conjecture 1.3) if we had not imposed the condition that the degree of the generic projection has to be strictly larger than 4. We refer to [Cat] for details and further information.

Notations

Varieties and morphisms

f	:	$X \to \mathbb{P}^2$	a generic projection from a smooth complex projective surface
f_{gal}	:	$X_{\text{gal}} \to \mathbb{P}^2$	the Galois closure of $f : X \to \mathbb{P}^2$
ℓ			a generic line in $\mathbb{P}^2$
$\mathbb{A}^2$	:=	$\mathbb{P}^2 - \ell$	the affine plane w.r.t. ℓ
X^{aff}	:=	$X - f^{-1}(\ell)$	the affine part of X w.r.t. ℓ
$X_{\text{gal}}^{\text{aff}}$	:=	$X_{\text{gal}} - {f_{\text{gal}}}^{-1}(\ell)$	the affine part of X_{gal} w.r.t. ℓ

Constructions in group theory

$[X, Y]$	subgroup generated by commutators $[x, y]$, $x \in X$, $y \in Y$
G^{ab}	abelianisation of a group G, i.e. the quotient $G/[G, G]$
$\mathcal{K}(G, n)$	the construction defined in Section 3.1
$\mathcal{E}(G, n)$	the construction defined in Section 3.1
X_G	the notation introduced in Section 3.3
$\mathcal{S}_n(d)$	the group defined in Section 5.1
$\widetilde{\mathcal{K}}(G, n)$	the construction defined in Section 5.3
$\widetilde{\mathcal{E}}(G, n)$	the construction defined in Section 5.3
$H_2(G)$	the second group homology with integral coefficients

Special groups

$\mathbb{Z}_n$	the cyclic group of order n
$\mathbb{Z}$	the infinite cyclic group
D_{2n}	the dihedral group of order $2n$
Π_g	the fundamental group of a smooth projective curve of genus $g \geq 1$
$\mathfrak{S}_n$	the symmetric group on n letters
$\mathfrak{S}_{n-1}^{(i)}$	the subgroup of $\mathfrak{S}_n$ fixing the letter i

Fundamental groups

$\pi_1^{\text{ét}}(Y)$	the étale or algebraic fundamental group
$\pi_1^{\text{top}}(Y)$	the topological fundamental group
$\pi_1^{\text{top}}(Y, G)$	the G-fundamental group defined in Section 4.3
$\pi_1^{\text{orb}}(Y, D_i, n_i)$	the orbifold fundamental group defined in Section 4.4

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