A BvM theorem for the Calderón problem with piecewise constant conductivities

Jan Bohr

Workshop on Theory for Scalable, Modern Statistical Methods

Bocconi University

6 April 2023

UNIVERSITÄT BONN

Overview

Research programme

Setting: **nonlinear inverse problems** on a parameter space Θ with $\dim \Theta = \infty$ or $\dim \Theta = D \to \infty$.

Goal: Theoretical guarantees for estimation and uncertainty quantification.

[Abraham, Altmeyer, B., Giordano, Kekkonen, Lassas, Monard, Nickl, Paternain, Siltanen, van de Geer, Wang, . . . (2016+)]

Research programme

Setting: **nonlinear inverse problems** on a parameter space Θ with $\dim \Theta = \infty$ or $\dim \Theta = D \to \infty$.

Goal: Theoretical guarantees for estimation and uncertainty quantification.

[Abraham, Altmeyer, B., Giordano, Kekkonen, Lassas, Monard, Nickl, Paternain, Siltanen, van de Geer, Wang, ... (2016+)]

- **Example:** In a Bayesian setting, we wish to approximate posterior measures by simpler ones (Gaussian, log-concave, ...).

Research programme

Setting: **nonlinear inverse problems** on a parameter space Θ with $\dim \Theta = \infty$ or $\dim \Theta = D \to \infty$.

Goal: Theoretical guarantees for estimation and uncertainty quantification.

[Abraham, Altmeyer, B., Giordano, Kekkonen, Lassas, Monard, Nickl, Paternain, Siltanen, van de Geer, Wang, . . . (2016+)]

- **Example:** In a Bayesian setting, we wish to approximate posterior measures by simpler ones (Gaussian, log-concave, ...).
- Possible for some **mildly ill-posed** inverse problems, provided the information operator is invertible/admits polynomial eigenvalue lower bounds.

Research programme

Setting: **nonlinear inverse problems** on a parameter space Θ with $\dim\Theta = \infty$ or $\dim\Theta = D \to \infty$.

Goal: Theoretical guarantees for estimation and uncertainty quantification.

[Abraham, Altmeyer, B., Giordano, Kekkonen, Lassas, Monard, Nickl, Paternain, Siltanen, van de Geer, Wang, ... (2016+)]

- **Example:** In a Bayesian setting, we wish to approximate posterior measures by simpler ones (Gaussian, log-concave, ...).
- Possible for some **mildly ill-posed** inverse problems, provided the information operator is invertible/admits polynomial eigenvalue lower bounds.
- For **severly ill-posed** problems, this hopeless for $D \to \infty$. Fix D?

Research programme

Setting: **nonlinear inverse problems** on a parameter space Θ with $\dim\Theta = \infty$ or $\dim\Theta = D \to \infty$.

Goal: Theoretical guarantees for estimation and uncertainty quantification.

[Abraham, Altmeyer, B., Giordano, Kekkonen, Lassas, Monard, Nickl, Paternain, Siltanen, van de Geer, Wang, ... (2016+)]

- **Example:** In a Bayesian setting, we wish to approximate posterior measures by simpler ones (Gaussian, log-concave, ...).
- Possible for some **mildly ill-posed** inverse problems, provided the information operator is invertible/admits polynomial eigenvalue lower bounds.
- For **severly ill-posed** problems, this hopeless for $D \to \infty$. Fix D?

Upshot

Even for extremely severe ill-posed problems, the statistical theory can behave well in a parametric setting.

The Calderón problem (1/2)

Electrical impedance tomography

- Infer **conductivity** inside body from **voltage/current measurements** at electrodes;
- applications: stroke identification, pulmonary monitoring, . . .

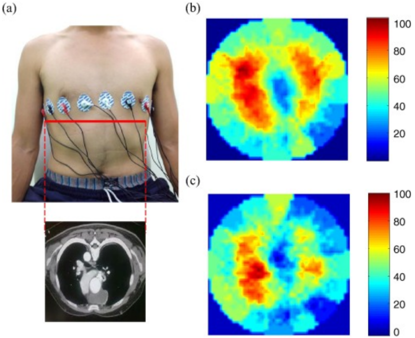


Figure: [Huang et. al. (2016)]

The Calderón problem (2/2)

- Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with smooth boundary.

The Calderón problem (2/2)

- Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with smooth boundary.
- For $\gamma \in L^\infty(\Omega, \mathbb{R})$ with $\operatorname{ess\,inf} \gamma > 0$ and $g \in H^{1/2}(\partial\Omega, \mathbb{R})$ there is a unique solution to

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{cases}$$

The Calderón problem (2/2)

- Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with smooth boundary.
- For $\gamma \in L^\infty(\Omega, \mathbb{R})$ with $\operatorname{ess\,inf} \gamma > 0$ and $g \in H^{1/2}(\partial\Omega, \mathbb{R})$ there is a unique solution to

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) & = & 0 & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega. \end{cases}$$

- The **Dirichlet-to-Neumann operator** of γ is defined by

$$\Lambda_\gamma g = \gamma \partial_\nu u.$$

This is a linear map $\Lambda_\gamma \colon H^{1/2} \to H^{-1/2}$, where $H^s = H^s(\partial\Omega, \mathbb{R})$.

The Calderón problem (2/2)

- Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with smooth boundary.
- For $\gamma \in L^\infty(\Omega, \mathbb{R})$ with $\operatorname{ess\,inf} \gamma > 0$ and $g \in H^{1/2}(\partial\Omega, \mathbb{R})$ there is a unique solution to

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) & = & 0 & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega. \end{cases}$$

- The **Dirichlet-to-Neumann operator** of γ is defined by

$$\Lambda_\gamma g = \gamma \partial_\nu u.$$

  This is a linear map $\Lambda_\gamma \colon H^{1/2} \to H^{-1/2}$, where $H^s = H^s(\partial\Omega, \mathbb{R})$.
- **Electrical Impedance Tomography (EIT):**

$$\gamma = \text{conductivity}, \quad \Lambda_\gamma \colon \text{voltage} \mapsto \text{current}$$

The Calderón problem (2/2)

- Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with smooth boundary.
- For $\gamma \in L^\infty(\Omega, \mathbb{R})$ with $\operatorname{ess\,inf} \gamma > 0$ and $g \in H^{1/2}(\partial\Omega, \mathbb{R})$ there is a unique solution to

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) & = & 0 & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega. \end{cases}$$

- The **Dirichlet-to-Neumann operator** of γ is defined by

$$\Lambda_\gamma g = \gamma \partial_\nu u.$$

  This is a linear map $\Lambda_\gamma \colon H^{1/2} \to H^{-1/2}$, where $H^s = H^s(\partial\Omega, \mathbb{R})$.
- **Electrical Impedance Tomography (EIT):**

$$\gamma = \text{conductivity}, \quad \Lambda_\gamma \colon \text{voltage} \mapsto \text{current}$$

The Calderón problem

Determine γ in the interior of Ω from measurements of Λ_γ at $\partial\Omega$.

Deterministic:

- For many classes $E \subset L^\infty(\Omega, \mathbb{R})$, we know **injectivity** of

$$\gamma \mapsto \Lambda_\gamma, \quad E \cap L^\infty_+(\Omega) \to \mathcal{B}(H^{1/2}, H^{-1/2}).$$

E.g.:

- $E = \{\text{piecewise analytic}\}$ [Kohn-Vogelius (1985)]
- $d \geq 3$: $E = C^2(\bar{\Omega})$ [Sylvester-Uhlmann (1987)]
- $d = 2$: $E = L^\infty(\Omega)$ [Nachman (1996), Astala-Päivärinta (2006)]

Deterministic:

- For many classes $E \subset L^\infty(\Omega, \mathbb{R})$, we know **injectivity** of

$$\gamma \mapsto \Lambda_\gamma, \quad E \cap L^\infty_+(\Omega) \to \mathcal{B}(H^{1/2}, H^{-1/2}).$$

  E.g.:
  - $E = \{\text{piecewise analytic}\}$ [Kohn-Vogelius (1985)]
  - $d \geq 3$: $E = C^2(\bar{\Omega})$ [Sylvester-Uhlmann (1987)]
  - $d = 2$: $E = L^\infty(\Omega)$ [Nachman (1996), Astala-Päivärinta (2006)]
- there are **reconstruction methods** (based on [Nachman (1988, 1996)]) that have been implemented numerically;

(Extremely) brief literature review

Deterministic:

- For many classes $E \subset L^\infty(\Omega, \mathbb{R})$, we know **injectivity** of

$$\gamma \mapsto \Lambda_\gamma, \quad E \cap L^\infty_+(\Omega) \to \mathcal{B}(H^{1/2}, H^{-1/2}).$$

  E.g.:
  - $E = \{\text{piecewise analytic}\}$ [Kohn-Vogelius (1985)]
  - $d \geq 3$: $E = C^2(\bar{\Omega})$ [Sylvester-Uhlmann (1987)]
  - $d = 2$: $E = L^\infty(\Omega)$ [Nachman (1996), Astala-Päivärinta (2006)]
- there are **reconstruction methods** (based on [Nachman (1988, 1996)]) that have been implemented numerically;
- however, typically the problem is **severly ill-posed** [Mandache, 2001]:

$$\|\gamma - \gamma'\|_\infty \leq \omega(\|\Lambda_\gamma - \Lambda_{\gamma'}\|) \quad \Rightarrow \quad \omega(t) \gtrsim |\ln(t)|^{-\sigma}$$

  (e.g. if $E = \{\gamma \in C^m(\bar{\Omega}) : \|\gamma\|_{C^m} \leq M\}$, $M, m \gg 1$)

Deterministic:

- For many classes $E \subset L^\infty(\Omega, \mathbb{R})$, we know **injectivity** of

$$\gamma \mapsto \Lambda_\gamma, \quad E \cap L^\infty_+(\Omega) \to \mathcal{B}(H^{1/2}, H^{-1/2}).$$

  E.g.:
  - $E = \{\text{piecewise analytic}\}$ [Kohn-Vogelius (1985)]
  - $d \geq 3$: $E = C^2(\bar{\Omega})$ [Sylvester-Uhlmann (1987)]
  - $d = 2$: $E = L^\infty(\Omega)$ [Nachman (1996), Astala-Päivärinta (2006)]
- there are **reconstruction methods** (based on [Nachman (1988, 1996)]) that have been implemented numerically;
- however, typically the problem is **severly ill-posed** [Mandache, 2001]:

$$\|\gamma - \gamma'\|_\infty \leq \omega(||\Lambda_\gamma - \Lambda_{\gamma'}\|) \quad \Rightarrow \quad \omega(t) \gtrsim |\ln(t)|^{-\sigma}$$

  (e.g. if $E = \{\gamma \in C^m(\bar{\Omega}) : \|\gamma\|_{C^m} \leq M\}$, $M, m \gg 1$)

Statistical:

- Non-parametric estimators typically converge **no better than logarithmically** in the inverse noise level [Abraham-Nickl (2019)] (Noise model: e.g. white noise in $\mathcal{B}_{\mathrm{HS}}$)

Question: How far can we push if we restrict to a parametric setting?

- Fix $E \subset L^\infty(\Omega)$ with $\dim E = D < \infty$.

Question: How far can we push if we restrict to a parametric setting?

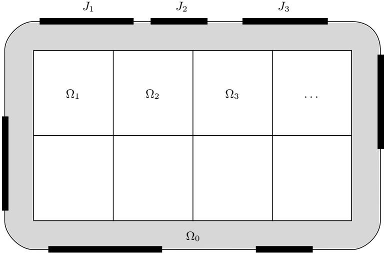
- Fix $E \subset L^\infty(\Omega)$ with $\dim E = D < \infty$.
- There is hope to set up a statistical experiment with measurements only at finitely many locations:

Theorem [Harrach (2019), Alberti-Santacesaria (2021)]

Inverse problems in a **finite dimensional unknown** can be solved stably with **finitely many noiseless measurements**, provided:

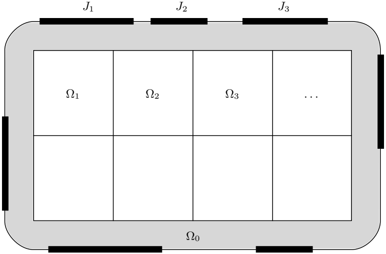
1. the nonlinear problem is injective;
2. its linearisation is injective;
3. some compactness properties are satisfied.

Suppose $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_D$ $(D \in \mathbb{N})$ is **known partition**, such as:



Piecewise constant Calderón problem (1/2)

Suppose $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_D$ $(D \in \mathbb{N})$ is **known partition**, such as:

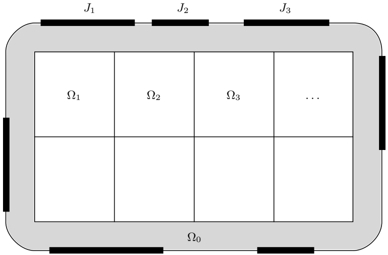


- We assume that the conductivity γ satisfies **known upper/lower bounds** $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$ and that it is piecewise constant:

$$\gamma = \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k}, \quad \theta \in \mathbb{R}^D$$

Piecewise constant Calderón problem (1/2)

Suppose $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_D$ $(D \in \mathbb{N})$ is **known partition**, such as:



- We assume that the conductivity γ satisfies **known upper/lower bounds** $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$ and that it is piecewise constant:

$$\gamma = \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k}, \quad \theta \in \mathbb{R}^D$$

- We measure Λ_γ at a finite set of electrodes $\mathbf{J} = \{J_1, \dots, J_M\}$, assuming

$$\Delta(\mathbf{J}) \ll 1, \qquad \text{where } \Delta(\mathbf{J}) = \left|\partial\Omega \setminus \bigcup_{k=1}^{M} J_k\right| + \sup_{k=1,\dots,M} \operatorname{diam} J_k.$$

Theorem

$$E_D := \left\{\gamma_\theta \in L^\infty(\Omega) : \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k},\ \theta \in \mathbb{R}^D\right\}$$

$$E'_D := \{\kappa \in L^\infty(\Omega) : \kappa + \mathbf{1}_{\Omega_0} \in E_D\}$$

On these spaces, the following properties hold true:

Theorem

$$E_D := \left\{ \gamma_\theta \in L^\infty(\Omega) : \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k},\ \theta \in \mathbb{R}^D \right\}$$

$$E'_D := \{ \kappa \in L^\infty(\Omega) : \kappa + \mathbf{1}_{\Omega_0} \in E_D \}$$

On these spaces, the following properties hold true:

(P1) $E_D \cap L^\infty_+(\Omega) \ni \gamma \to \Lambda_\gamma$ *is injective;*

Piecewise constant Calderón problem (2/2)

Theorem

$$E_D := \left\{\gamma_\theta \in L^\infty(\Omega) : \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k},\ \theta \in \mathbb{R}^D\right\}$$

$$E'_D := \{\kappa \in L^\infty(\Omega) : \kappa + \mathbf{1}_{\Omega_0} \in E_D\}$$

On these spaces, the following properties hold true:

(P1) $E_D \cap L^\infty_+(\Omega) \ni \gamma \to \Lambda_\gamma$ *is injective;*

(P2) *for all* $\gamma \in E_D \cap L^\infty_+(\Omega)$, *the differential* $E'_D \ni \kappa \mapsto d\Lambda_\gamma(\kappa)$ *is injective;*

Theorem

$$E_D := \left\{\gamma_\theta \in L^\infty(\Omega) : \gamma_\theta \equiv \mathbf{1}_{\Omega_0} + \sum_{k=1}^{D} \theta_k \mathbf{1}_{\Omega_k},\ \theta \in \mathbb{R}^D\right\}$$

$$E'_D := \{\kappa \in L^\infty(\Omega) : \kappa + \mathbf{1}_{\Omega_0} \in E_D\}$$

On these spaces, the following properties hold true:

(P1) $E_D \cap L^\infty_+(\Omega) \ni \gamma \to \Lambda_\gamma$ *is injective;*

(P2) *for all* $\gamma \in E_D \cap L^\infty_+(\Omega)$*, the differential* $E'_D \ni \kappa \mapsto d\Lambda_\gamma(\kappa)$ *is injective;*

(P3) *for all* $\gamma \in E_D \cap L^\infty_+(\Omega)$ *and* $\kappa \in E'_D$*, the operator* $d\Lambda_\gamma(\kappa)$ *is smoothing and* $(\gamma, \kappa) \mapsto \|d\Lambda_\gamma(\kappa)\|_{H^s \to H^t}$ *is locally bounded for any* $s, t \in \mathbb{R}$*.*

- Lipschitz stability (linear & non-linear) comes for free [Bourgeois (2013)].

From noiseless to noisy measurements (1/2)

Given a set of electrodes $\mathbf{J} = \{J_1, \ldots, J_M\}$, and with $\tilde{\Lambda}_\gamma = \Lambda_\gamma - \Lambda_{\mathbf{1}}$, define:

$$\begin{aligned} \Theta &:= \{\theta \in \mathbb{R}^D : \gamma_{\min} \leq \theta_i \leq \gamma_{\max},\ i = 1, \ldots, D\} \\ G_\theta^{ij} &:= \frac{1}{(|J_i||J_j|)^{1/2}} \langle \tilde{\Lambda}_{\gamma_\theta} \mathbf{1}_{J_i}, \mathbf{1}_{J_j} \rangle_{L^2(\partial\Omega)}, \quad 1 \leq i, j, \leq M,\ \theta \in \Theta, \end{aligned}$$

From noiseless to noisy measurements (1/2)

Given a set of electrodes $\mathbf{J} = \{J_1, \dots, J_M\}$, and with $\tilde{\Lambda}_\gamma = \Lambda_\gamma - \Lambda_{\mathbf{1}}$, define:

$$\begin{aligned} \Theta &:= \{\theta \in \mathbb{R}^D : \gamma_{\min} \leq \theta_i \leq \gamma_{\max},\ i = 1, \dots, D\} \\ G_\theta^{ij} &:= \frac{1}{(|J_i||J_j|)^{1/2}} \langle \tilde{\Lambda}_{\gamma_\theta} \mathbf{1}_{J_i}, \mathbf{1}_{J_j} \rangle_{L^2(\partial\Omega)}, \quad 1 \leq i, j, \leq M,\ \theta \in \Theta, \end{aligned}$$

Theorem (Finitely many noiseless measurements)

If $\Delta(\mathbf{J}) \leq \delta(E_D, \gamma_{\min}, \gamma_{\max})$, then the following map is injective:

$$\Theta \to \mathbb{R}^{M \times M}, \quad \theta \mapsto G_\theta = (G_\theta^{ij} : 1 \leq i, j \leq M),$$

It obeys a Lipschitz stability estimate and has derivative of full rank.

From noiseless to noisy measurements (1/2)

Given a set of electrodes $\mathbf{J} = \{J_1, \ldots, J_M\}$, and with $\tilde{\Lambda}_\gamma = \Lambda_\gamma - \Lambda_{\mathbf{1}}$, define:

$$\begin{aligned} \Theta &:= \{\theta \in \mathbb{R}^D : \gamma_{\min} \leq \theta_i \leq \gamma_{\max},\ i = 1, \ldots, D\} \\ G_\theta^{ij} &:= \frac{1}{(|J_i||J_j|)^{1/2}} \langle \tilde{\Lambda}_{\gamma_\theta} \mathbf{1}_{J_i}, \mathbf{1}_{J_j} \rangle_{L^2(\partial\Omega)}, \quad 1 \leq i, j, \leq M,\ \theta \in \Theta, \end{aligned}$$

Theorem (Finitely many noiseless measurements)

If $\Delta(\mathbf{J}) \leq \delta(E_D, \gamma_{\min}, \gamma_{\max})$, then the following map is injective:

$$\Theta \to \mathbb{R}^{M \times M}, \quad \theta \mapsto G_\theta = (G_\theta^{ij} : 1 \leq i, j \leq M),$$

It obeys a Lipschitz stability estimate and has derivative of full rank.

- Direct consequence of [Harrach (2019), Alberti-Santacesaria (2021)];

From noiseless to noisy measurements (1/2)

Given a set of electrodes $\mathbf{J} = \{J_1, \ldots, J_M\}$, and with $\tilde{\Lambda}_\gamma = \Lambda_\gamma - \Lambda_{\mathbf{1}}$, define:

$$\begin{aligned} \Theta &:= \{\theta \in \mathbb{R}^D : \gamma_{\min} \le \theta_i \le \gamma_{\max},\ i = 1, \ldots, D\} \\ G_\theta^{ij} &:= \frac{1}{(|J_i||J_j|)^{1/2}} \langle \tilde{\Lambda}_{\gamma_\theta} \mathbf{1}_{J_i}, \mathbf{1}_{J_j} \rangle_{L^2(\partial\Omega)}, \quad 1 \le i, j, \le M,\ \theta \in \Theta, \end{aligned}$$

Theorem (Finitely many noiseless measurements)

If $\Delta(\mathbf{J}) \le \delta(E_D, \gamma_{\min}, \gamma_{\max})$, then the following map is injective:

$$\Theta \to \mathbb{R}^{M \times M}, \quad \theta \mapsto G_\theta = (G_\theta^{ij} : 1 \le i, j \le M),$$

It obeys a Lipschitz stability estimate and has derivative of full rank.

- Direct consequence of [Harrach (2019), Alberti-Santacesaria (2021)];
- δ depends (fairly explicitly) on Lipschitz constants, and these grow exponentially in D.

From noiseless to noisy measurements (2/2)

Statistical experiment

Let $\mathcal{X} = \{1, \dots, M\}$ and $\lambda =$ *counting measure* and recast G as follows:

$$\mathcal{G} \colon \Theta \mapsto L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \quad \mathcal{G}_\theta(x) = (G_\theta^{x,j} : j = 1, \dots, M)$$

Let $P_\theta = \mathrm{Law}(Y, X)$, where (Y, X) follows the *regression equation*

$$Y = \mathcal{G}_\theta(X) + \epsilon, \quad X \sim \lambda, \ \epsilon \sim \mathcal{N}_M(0, I).$$

Put $P_\theta^N = P_\theta \otimes \cdots \otimes P_\theta$ and $\mathcal{P}_N = \{P_\theta^N : \theta \in \Theta\}$.

From noiseless to noisy measurements (2/2)

Statistical experiment

Let $\mathcal{X} = \{1, \dots, M\}$ and $\lambda =$ *counting measure* and recast G as follows:

$$\mathcal{G} \colon \Theta \mapsto L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \quad \mathcal{G}_\theta(x) = (G_\theta^{x,j} : j = 1, \dots, M)$$

Let $P_\theta = \mathrm{Law}(Y, X)$, where (Y, X) follows the *regression equation*

$$Y = \mathcal{G}_\theta(X) + \epsilon, \quad X \sim \lambda, \ \epsilon \sim \mathcal{N}_M(0, I).$$

Put $P_\theta^N = P_\theta \otimes \cdots \otimes P_\theta$ and $\mathcal{P}_N = \{P_\theta^N : \theta \in \Theta\}$.

- **Interpretation:** We choose an electrode $J \in \mathbf{J}$ at random and apply a standard voltage. The resulting current is then measured at all electrodes, corrupted by Gaussian noise. Repeat this N-times.

From noiseless to noisy measurements (2/2)

Statistical experiment

Let $\mathcal{X} = \{1, \dots, M\}$ and $\lambda =$ *counting measure* and recast G as follows:

$$\mathcal{G} \colon \Theta \mapsto L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \quad \mathcal{G}_\theta(x) = (G_\theta^{x,j} : j = 1, \dots, M)$$

Let $P_\theta = \mathrm{Law}(Y, X)$, where (Y, X) follows the *regression equation*

$$Y = \mathcal{G}_\theta(X) + \epsilon, \quad X \sim \lambda, \ \epsilon \sim \mathcal{N}_M(0, I).$$

Put $P_\theta^N = P_\theta \otimes \cdots \otimes P_\theta$ and $\mathcal{P}_N = \{P_\theta^N : \theta \in \Theta\}$.

- **Interpretation:** We choose an electrode $J \in \mathbf{J}$ at random and apply a standard voltage. The resulting current is then measured at all electrodes, corrupted by Gaussian noise. Repeat this N-times.
- The experiment $\mathcal{P} = \mathcal{P}_1$ is *differentiable in quadratic mean* and has a well-defined **information matrix**

$$\mathbb{N}_\theta \equiv d\mathcal{G}_\theta^* d\mathcal{G}_\theta, \quad \theta \in \Theta.$$

From noiseless to noisy measurements (2/2)

Statistical experiment

Let $\mathcal{X} = \{1, \dots, M\}$ and $\lambda =$ *counting measure* and recast G as follows:

$$\mathcal{G} \colon \Theta \mapsto L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \quad \mathcal{G}_\theta(x) = (G^{x,j}_\theta : j = 1, \dots, M)$$

Let $P_\theta = \mathrm{Law}(Y, X)$, where (Y, X) follows the *regression equation*

$$Y = \mathcal{G}_\theta(X) + \epsilon, \quad X \sim \lambda, \ \epsilon \sim \mathcal{N}_M(0, I).$$

Put $P^N_\theta = P_\theta \otimes \cdots \otimes P_\theta$ and $\mathcal{P}_N = \{P^N_\theta : \theta \in \Theta\}$.

- **Interpretation:** We choose an electrode $J \in \mathbf{J}$ at random and apply a standard voltage. The resulting current is then measured at all electrodes, corrupted by Gaussian noise. Repeat this N-times.
- The experiment $\mathcal{P} = \mathcal{P}_1$ is *differentiable in quadratic mean* and has a well-defined **information matrix**

$$\mathbb{N}_\theta \equiv d\mathcal{G}^*_\theta d\mathcal{G}_\theta, \quad \theta \in \Theta.$$

Theorem

The information matrix $\mathbb{N}_\theta \in \mathbb{R}^{D \times D}$ is invertible for all $\theta \in \Theta \backslash \partial\Theta$. □

Consequences

Now we are in the classical setting of [Asymptotic Statistics, van der Vaart (1998)], with all the nice consequences:

Consequences

Now we are in the classical setting of [Asymptotic Statistics, van der Vaart (1998)], with all the nice consequences:

1. **Bernstein-von-Mises theorem:** If $\Pi = \pi(\theta)d\theta$ is a sufficiently regular prior on Θ (e.g $\pi \equiv 1/|\Theta|$), $\theta_0 \in \Theta\backslash\partial\Theta$ and $\vartheta \sim \Pi(\cdot|(Y_i, X_i)_{i=1}^N)$ is a posterior draw, then:

$$\left\|\mathrm{Law}\left(\sqrt{N}(\vartheta - \bar{\theta}_N)\middle|(Y_i, X_i)_{i=1}^N\right) - \mathcal{N}_D(0, \mathbb{N}_{\theta_0}^{-1})\right\|_{\mathrm{TV}} \to 0 \text{ in } P_{\theta_0}^N.$$

Here $\bar{\theta}_N = \mathbb{E}[\vartheta|(Y_i, X_i)_{i=1}^N]$ is the posterior mean.

Consequences

Now we are in the classical setting of [Asymptotic Statistics, van der Vaart (1998)], with all the nice consequences:

1. **Bernstein-von-Mises theorem:** If $\Pi = \pi(\theta)d\theta$ is a sufficiently regular prior on Θ (e.g $\pi \equiv 1/|\Theta|$), $\theta_0 \in \Theta \backslash \partial\Theta$ and $\vartheta \sim \Pi(\cdot|(Y_i, X_i)_{i=1}^N)$ is a posterior draw, then:

$$\left\| \mathrm{Law}\left(\sqrt{N}(\vartheta - \bar{\theta}_N) \middle| (Y_i, X_i)_{i=1}^N \right) - \mathcal{N}_D(0, \mathbb{N}_{\theta_0}^{-1}) \right\|_{\mathrm{TV}} \to 0 \text{ in } P_{\theta_0}^N.$$

   Here $\bar{\theta}_N = \mathbb{E}[\vartheta|(Y_i, X_i)_{i=1}^N]$ is the posterior mean.

2. **Optimal asymptotic minimax variance:** For all estimator sequences $(T_N : N \in \mathbb{N})$ we have

$$\lim_{\delta \to 0} \liminf_{N \to \infty} \sup_{\|\theta - \theta_0\| < \delta} \mathrm{Cov}_\theta^N \left[\sqrt{N}(T_N - \theta) \right] \geq \mathbb{N}_{\theta_0}^{-1},$$

   and $T_N = \bar{\theta}_N$ achieves this lower bound ($\rightsquigarrow$ no analogue of this for Nachman's reconstruction procedure).

3. ...

Concluding remarks:

- The Calderón problem is just an example;

Concluding remarks:

- The Calderón problem is just an example;
- essentially all nonlinear inverse problems for which the **forward map** and its **linearisation are injective** can be fitted into the classical parametric setting like this;

Concluding remarks:

- The Calderón problem is just an example;
- essentially all nonlinear inverse problems for which the **forward map** and its **linearisation are injective** can be fitted into the classical parametric setting like this;
- only relevant for **severaly ill-posed** problems — in the **mildly ill-posed** case there is also a satisfactory non-parametric theory.

Concluding remarks:

- The Calderón problem is just an example;
- essentially all nonlinear inverse problems for which the **forward map** and its **linearisation are injective** can be fitted into the classical parametric setting like this;
- only relevant for **severaly ill-posed** problems — in the **mildly ill-posed** case there is also a satisfactory non-parametric theory.

Thank you!