Euler equations and harmonic analysis

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Abstract

These few pages are notes written for a talk given at the PhD student seminar of l'*Institut Fourier* in Grenoble. We will first shortly discuss the (homogeneous) Euler equations and the type of problems that can arise when studying well-posedness of a PDE system. Afterwards, we will focus on th 2D case and adress the question of existence of solutions for all times by using a fairly recent method (2005) involving Littlewood-Paley analysis.

Forwards

As mentioned above, these are notes written for a one-hour long talk and must be treated as such. Even though they probably contain more details than will be given orally, they remain incomplete. However, we hope that the accompanying references will be sufficient for any reasonably interested reader to fill in the gaps.

1 Introduction

Th Euler equations is the second (system of) Partial Differential Equations ever to be written. They were derived in 1757 by the eponymous mathematician to describe the dynamics of inviscid fluids. For homogeneous flows, they read

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \\ \mathrm{div}(u) = 0, \end{cases} \tag{1}$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \longrightarrow \mathbb{R}^d$ is a the velocity field and $p : \mathbb{R}_t \times \mathbb{R}^d_x \longrightarrow \mathbb{R}$ is the pressure of the fluid, which evolves in the whole d-dimensional space $\mathbb{R}^d$.

The so-called *convective term* $(u \cdot \nabla)u$ is written

$$(u \cdot \nabla) u = \sum_k u_k \partial_k u \tag{2}$$

and makes the Euler equations very similar to the more familiar transport equations.

Although this system has been extensively studied by generations of mathematicians, many open questions remain attached to these equations, notably in three dimensions. For instance, in that case, it is still unknown if there exists global smooth solutions, *i.e.* solutions which are defined for all times.

Ou purpose in this text is to explore some of the theory pertaining to the 2D Euler equations. We will (sketchily) prove existence of global solutions by using a method due to T. Hmidi and S. Kernaani ([5], 2005), involving powerful Fourier analisys tools which are used to solve many PDE problems.

2 General remarks on the equations

2.1 Physical remarks

The Euler equations model perfect inviscid fluids which have constant density $\bar{\rho}$, which we set to unit value for the sake of simplicity, as well as constant temperature. This approximation defines the notion of *perfect homogeneous fluid*, and is fairly accurate for most fluids at human scale: the air in a room, water in a glass, *etc.*

The first equation in the system is a momentum balance. It states that the (local) variation in momentum $\partial_t u$ is due to convection in the fluid (the $(u \cdot \nabla)u$ term) and to pressure forces.

The second equation is simply mass conservation, or equivalently, since the fluid is assumed to have constant density, volume conservation.

The Euler equations describe *non-viscous* fluids, so that there is no loss of energy due to internal friction. The fluid is only described in terms of Newtonian mechanics, with no regards to thermodynamics. The consequence of this is that the Euler equations are reversible: any solution $u(t,x)$ gives rise to another solution $u(-t,x)$ by time reversal. This is no longer the case if we add a diffusive term[1] in the momentum balance

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \Delta u \\ \mathrm{div}(u) = 0, \end{cases} \tag{3}$$

which makes the equations much more similar to the heat equation. For this variant, the Navier-Stokes equations, it is a well-known fact that the equations are not reversible[2].

Finally, as we will see in the next paragraph, the pressure of the fluid can fully be expressed as a function of the velocity. This means that the fluid satisfies no state equation $p = F(\rho)$ which would link the pressure to the (constant) density $\rho = \bar{\rho}$

2.2 The pressure term

Strictly speaking, the Euler equations form a system of PDE with *two* unknowns: the velocity field u and the pressure p. However, as shown in this section, the pressure is entirely given by knowledge of the velocity field.

Taking the divergence of the momentum equation, we see that p solves an elliptic problem

$$-\Delta p = \sum_{j,k} \partial_k \partial_j (u_k u_j) \tag{4}$$

whose solution is given (for example) by means of Fourier multipliers

$$\nabla p = \nabla(-\Delta)^{-1} \sum_{j,k} \partial_k \partial_j (u_k u_j), \tag{5}$$

or, in other words,

$$\widehat{\nabla p}(\xi) = -i\xi \sum_{j,k} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{u_j u_k}(\xi). \tag{6}$$

The consequence of these computations is that (1) can be considered as a problem with a single unknown. One way to say this is that the pressure term can either be replaced by (5).

[1] There is generally very little theoretical justification for the exact form of the viscosity term. It is mainly chosen because the fluid studied is assumed to be Newtonian. In other words, it satisfies a phenomenological law.

[2] For example, it has been proved that the kinetic energy of weak solutions of the Navier-Stokes system tends to zero in the limit $t \to +\infty$.

Alternately, we can say that the momentum equation is written up to a gradient summand: we seek u such that there exists a function p satisfying the momentum equation. Therefore, this equation can be written

$$\partial_t u + (u \cdot \nabla)u = 0 \quad \big(\text{modulo gradients}\big). \tag{7}$$

The advantage of this last perspective is that it allows the pressure term to be understood as a Lagrange multiplier associated with the incompressibility constraint $\operatorname{div}(u) = 0$.

3 Toy model: Burgers equation

To better understand the kind of problems the Euler equations can cause, we study a simpler model[3], in one dimension of space: the Burgers equation

$$\partial_t u + \frac{1}{2}\partial_x \left(u^2\right) = 0. \tag{8}$$

3.1 Explicit solution

The adavntage of (8) is that it is one the very few PDEs which can be (more or less) explicitly solved. We introduce the characteristic curves: if u is a solution, then a characteristic is a function $x(t)$ which we chose so that $u\big(t, x(t)\big)$ is independent of time. This happens provided that

$$\frac{\mathrm{d}}{\mathrm{d}t} u\big(t, x(t)\big) = \partial_t u\big(t, x(t)\big) + x'(t)\partial_x u\big(t, x(t)\big) = 0. \tag{9}$$

As u solves (8), this quantity will obviously be zero if $x(t)$ solves the following ODE problem:

$$\begin{cases} x'(t) = u\big(t, x(t)\big) \\ x(0) = x. \end{cases} \tag{10}$$

In that case, $x(t)$ is a characteristic curve and the quantity $u\big(t, x(t)\big)$ is constant and equal to its value at time $t = 0$, that is $u(0, x) := u_0(x)$. This, in turn, enables us to compute explicitly the solution: as we have

$$x'(t) = u\big(t, x(t)\big) = \text{Cst} = u_0(x), \tag{11}$$

the characteristic is a straight line $x(t) = x + tu_0(x)$ and the solution of (8) must satisfy

$$u\big(t, x + tu_0(x)\big) = u_0(x). \tag{12}$$

This provides an implicit equation solved by u, which we can find as long as the function $x \mapsto x + tu_0(x)$ is invertible.

3.2 Different types of solutions

The solution given by (12) is as smooth as the initial datum u_0 is, as long as the solution exists, *i.e.* as long as the function

$$\phi_t(x) = x + tu_0(x) \tag{13}$$

is smoothly invertible. This is always the case if t is small enough. The inverse function ϕ_t^{-1} will cease to be smooth if there is a point $x_0 \in \mathbb{R}$ such that the derivative $\phi_t'(x_0)$ is zero. At that

[3]Incompressible fluids in one dimensions are not of extreme interest.

point, the solution we have constructed in (12) ceases to exist because the derivative $\partial_x u$ explodes in the L^∞ norm. This *finite time explosion* occurs at the instant

$$T^* = \frac{-1}{\min_{x\in\mathbb{R}} u_0'(x)}. \tag{14}$$

To recover global solutions (defined for all times), we must somehow allow for *discontinuous* solutions. This is where the notion of weak solution is involved. We say that $u \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x)$ is a weak solution of (8) with initial data u_0 is it solves the problem in the sense of distributions: for all $\varphi \in C^1(\mathbb{R}_t \times \mathbb{R}_x)$ with compact support, we have

$$\int\int \left\{ u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi \right\} \mathrm{d}x\,\mathrm{d}t + \int \phi(0,x)u_0(x)\,\mathrm{d}x = 0. \tag{15}$$

Such weak solutions do indeed exist for all times. Before time T^* they coincide with the smooth solution (12) and form a shockwave at time T^* which propagates afterwards in the solution.

However, the major problem of these weak solutions is that they are not unique. More precisely, for a fairly large set of initial data u_0, there are infinitely many weak solutions of (8).

Generally speaking, this is what tends to happen in the study of PDEs:

Smooth solutions which solve a PDE problem usually exist on (possibly) *finite* time intervalls. These smooth solutions are generally unique once (smooth) initial data is given. Proving that they are global is often evry difficult.

Weak solutions are usually global[4], but are not regular enough that uniqueness can be proved (easily).

A PDE problem is said to be *globally well-posed* if there exists a class of solutions in which there exists a unique (global) solution given appropriate initial data.

In the rest of this text, we show why the 2D Euler equations are in fact globally well-posed.

Remark 3.1. Burgers equation is a part of a more general class of problems, *scalar conservation laws*, which have been intensively studied. For example, a well-posedness theorem is in fact available in a special class of *entropy* solutions. See Chapter 11 of [4] (pp. 609-658) for an introduction to these equations.

4 Well-posedness for 2D Euler

In all that follows, we restrict ourselves to the 2D Euler equations, of which we will show global well-posedness. Most of what we present here can be found in [1], in Chapter 3, Section 3.2.2 and Chapter 7, Section 7.2.2.

4.1 Approximate system

The goal of this section is to give a vague idea of how to contruct of a PDE problem using *a priori* estimates. We refer to any textbook on PDE for more details. Concerning Euler equation, the 7th chapter of [1] has complete proofs of the contents of the following.

[4]However, in many cases, the mere existence of global weak solutions is a very hard problem. For instance, it is still unknown if the ideal MHD system has or not global solutions of any kind with general initial data (even in 2D!).

The main idea of the proof is to construct a sequence of (approximate) solutions which satisfy an approximate system, namely[5]

$$
\begin{cases} \partial_t u_{n+1} + (u_n \cdot \nabla) u_{n+1} + \nabla p_n = 0 \\ \operatorname{div}(u_{n+1}) = 0 \end{cases} \tag{16}
$$

with initial data $u_n(0) = S_n u_0$, for some regularizing operator S_n. Next, we seek uniform bounds in some Banach space X

$$
\forall n \geq 0, \qquad \|u_n\|_X \leq C \tag{17}
$$

in order to extract a weakly convergeing subsequence. If the space X has a strong enough topology, we can prove convergence of the sequence to a solution of (1).

Therefore, the main objective is finding good enough estimates (called *a priori estimates*) which (approximate) solutions of (1) satisfy. For example, we have seen that solutions satisfy the energy conservation law

$$
\forall t, \qquad \|u(t)\|_{L^2} = \|u_0\|_{L^2} \tag{18}
$$

so that (approximate) solutions will be bounded in the space $L^\infty_t(L^2_x)$. However, the $L^\infty(L^2)$ topology is not strong enough to show that the weak limit of the sequence (u_n) is indeed a solution of (1).

4.2 Vorticity for 2D Euler equations

The key feature for showing good *a priori* estimates for the 2D Euler equations is the vorticity ω, which is defined as the curl of the velocity field

$$
\omega = \partial_1 u_2 - \partial_1 u_2. \tag{19}
$$

The reason why the curl is an interesting quantity is that, in the two dimensionnal setting, it satisfies a simple transport equation

$$
\partial_t \omega + u \cdot \nabla \omega = 0. \tag{20}
$$

This equation can be derived from the momentum equation by taking its curl. The pressure term dissapears because of the Schwarz theorem, and algebraic cancellations specific to the 2D setting do the rest.

Giving estimates for ω is very easy, as transport equations preserve L^p norms. In the case of L^∞, for example, introduce the characteristic curves, which solve the ODE system

$$
\begin{cases} x'(t) = u\big(t, x(t)\big) \\ x(0) = x. \end{cases} \tag{21}
$$

so that the solution ω must satisfy $\omega\big(t, x(t)\big) = \omega_0(x)$. From that, we see that $\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$ at all times t.

To benefit from this estimate, we must write u as a function of the vorticity. Similarly to what we had with the pressure in (5), we have to solve an elliptic equation.

As the flow is divergence-free $\operatorname{div}(u) = 0$, we can introduce a stream function ϕ such that $u = \nabla^\perp \phi = (-\partial_2 \phi, \partial_1 \phi)$, which solves $\Delta \phi = \omega$. The velocity is written as

$$
u = -\nabla^\perp (-\Delta)^{-1} \omega, \tag{22}
$$

[5]This system is linear with respect to u_{n+1} so it is no problem to construct u_{n+1}. See [1], Chapter 3 for more on transport equations.

and all that remains is to study the properties of the opearator $\nabla^\perp(-\Delta)^{-1}$, which roughly acts as a derivative of order -1.

Unfortunately, things are not so simple. For example, when evaluating the derivative ∇u, one must consider the zero-order operator $\nabla\nabla^\perp(-\Delta)^{-1}$, which, unlike "nice" Fourier multipliers, cannot be defined as a convolution operator, because its kernel is not integrable!

One way of proceeding is to fall back on the Calderòn-Zygmund theory of Singular Integral Operators (SIOs for short), which states that the operator $\omega \mapsto \nabla u$ is continuous on every L^p space with $1 < p < +\infty$. However, this line of proof is terribly complicated, on top of forcing us to leave aside the endpoint cases $p = 1$ and $p = +\infty$.

Instead, we present an easier approach, which relies on a dyadic decomposition of the frequency space: Littlewood-Paley analysis.

Remark 4.1. For more on the SIO approach of the Euler problem, the interested reader may study [2] (Chapter 8 deals with weak solutions).

4.3 Littlewood-Paley decomposition

This section briefly presents some of the main ideas of Littlewood-Paley theory. We refer to [1] (notably Chapter 2) for a thorough introduction.

The main idea of the dyadic Littlewood-Paley decomposition is to isolate frequencies that have same order of magnitude, in order to treat them separately. With that in mind, we equip ourselves with a partition of the space: smooth nonnegative functions $\varphi_{-1}, \varphi_0, \varphi_1, \varphi_2, ...$ such that

1. we have $\sum_{j\geq -1} \varphi_j(\xi) = 1$,
2. the function φ_{-1} is supported in a ball $B(0, 2)$,
3. every other function φ_j is supported in an annulus $\mathcal{A}_j = \{2^j \leq |\xi| \leq 2^{j+2}\}$.

Next, we define the *dyadic blocks* Δ_j, frequency localizing operators, by their Fourier transform:

$$\forall f, \qquad \widehat{\Delta_j f}(\xi) = \varphi_j(\xi)\widehat{f}(\xi). \tag{23}$$

Because the sum of the φ_j is one, we see that any function f can be written as a sum

$$f = \sum_{j\geq -1} \Delta_j f. \tag{24}$$

This is called the Littlewood-Palez decomposition of f.

The main features of the Littlewood-Paley decomposition lie in the fact that differentiation broadly acts on the dyadic block $\Delta_j f$ as a dilation of order 2^j.

Lemma 4.2. *The dyadic blocks have the following useful properties: let $p \in [1, +\infty]$,*

1. *The operators Δ_j are bounded on every Lebesgue space L^p, for all $j \geq -1$ and their operator norm $\|\Delta_j\|_{\mathcal{L}(L^p)} \leq C$ is bounded independently of p and j.*
2. *(Bernstein inequality) For all $j \neq -1$, we have*

$$\forall f, \qquad C^{-1}2^j\|\Delta_j f\|_{L^p} \leq \|\Delta_j \nabla f\|_{L^p} \leq C2^j\|\Delta_j f\|_{L^p}, \tag{25}$$

for some constant C which depends only on the decomposition $(\varphi_j)_j$.

Littlewood-Paley theory is a recurrent tool in the analysis of PDE as it provides (amongst many other things) embedding theorems for spaces of functions, characterization of these spaces and systematic methods for dealing with products of functions (see [1] for numerous applications).

4.4 Linear estimate for the vorticity

In this section, we explain (very sketchily) how Littlewood-Paley decomposition can be used to obtain a linear estimate for the transport equation solved by ω. As we will see, this provides good *a priori* estimates on u.

The first step of our argument is to decompose the vorticity as a sum of functions ω_k which solve the linear transport equation[6]

$$\begin{cases} \partial_t \omega_k + u \cdot \nabla \omega_k = 0 \\ \omega_k(t=0) = \Delta_k \omega_0. \end{cases} \tag{26}$$

Because the transport equation is linear with respect to ω, and since we have $\sum_k \Delta_k \omega_0 = \omega_0$, we see that the vorticity can be written

$$\omega = \sum_{k \geq -1} \omega_k. \tag{27}$$

Therefore, we get,

$$\big\| \omega(t) \big\|_{L^\infty} \leq \sum_{j \geq -1} \big\| \Delta_j \omega(t) \big\|_{L^\infty} \leq \sum_{j,k \geq -1} \big\| \Delta_j \omega_k(t) \big\|_{L^\infty}. \tag{28}$$

Now, we separate this sum into two parts, according to whether j and k are close or not. Let N be in integer whose value we will fix later. We have

$$\sum_{j,k \geq -1} \| \Delta_j \omega_k(t) \|_{L^\infty} = \sum_{|j-k| \leq N} \big\| \Delta_j \omega_k(t) \big\|_{L^\infty} + \sum_{|j-k| > N} \big\| \Delta_j \omega_k(t) \big\|_{L^\infty}. \tag{29}$$

On the one hand, we use zero-order estimates to bound the first sum: recalling that the transport equation conserves L^∞ norms and that the dyadic blocks Δ_j are continuous operators on that same space, we see that

$$\begin{aligned} \sum_{|j-k| \leq N} \big\| \Delta_j \omega_k(t) \big\|_{L^\infty} &\leq C \sum_{|j-N| \leq N} \big\| \omega_k(t) \big\|_{L^\infty} \\ &= C \sum_{|j-N| \leq N} \big\| \omega_k(t=0) \big\|_{L^\infty} = C \sum_{|j-N| \leq N} \big\| \Delta_k \omega_0 \big\|_{L^\infty} = CN \sum_{k \geq -1} \big\| \Delta_k \omega_0 \big\|_{L^\infty}. \end{aligned} \tag{30}$$

Prompted by this inequaity, we define a new norm: for any appropriate $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, set

$$\|f\|_B := \sum_{k \geq -1} \big\| \Delta_k f \big\|_{L^\infty}. \tag{31}$$

Remark 4.3. Although we will not explore this topic further, it is worthy to note that the Banach space associated to the norm $\| \, . \, \|_B$ is a one amongst many *Besov spaces*, which can also be defined as interpolation between Sobolev spaces.

What we have obtained for the sum $|j-k| \leq N$ is the inequality

$$\sum_{|j-k| \leq N} \big\| \Delta_j \omega_k(t) \big\|_{L^\infty} \leq CN \big\| \omega_0 \|_B. \tag{32}$$

[6] Here, we implicitly use the fact that the transport equation has a unique solution in the Banach space in which ω lies. Chapter 3 of [1] is devoted to these equations and their well-posedness in a very general class of functions.

On the other hand, to deal with the sum $|j-k|>N$, we need a higher order estimate. For example, it is fairly straightforward to find a order one estimate by differentiating (4.2). We have

$$\partial_t \partial_j \omega + u \cdot \nabla \partial_j \omega = -\partial_j u \cdot \nabla \omega. \tag{33}$$

Introducing characteristics as before, we immediately get the estimate

$$\|\nabla \omega\|_{L^\infty} \leq \|\nabla \omega_0\|_{L^\infty} + \int_0^t \big\|\nabla u(s)\big\|_{L^\infty} \big\|\nabla \omega(s)\big\|_{L^\infty} \mathrm{d}s \tag{34}$$

and Grönwall's lemma gives the exponential estimate

$$\|\nabla \omega\|_{L^\infty} \leq \|\nabla \omega_0\|_{L^\infty} \exp\left(\int_0^t \big\|\nabla u(s)\big\|_{L^\infty} \mathrm{d}s\right). \tag{35}$$

Now, for the sake of conciseness, we skip a few details, referring the interested reader to [1] for a complete proof (Chapter 3, Section 3.2.2, pp.135-136). Roughly, what happens is that we use an estimate of order ϵ for the vorticity and get the upper bound[7]

$$\sum_{|j-k|>N} \big\|\Delta_j \omega_k(t)\big\|_{L^\infty} \leq C 2^{-N\epsilon} \|\omega_0\|_B \exp\left(C \int_0^t \big\|\nabla u(s)\big\|_{L^\infty} \mathrm{d}s\right). \tag{36}$$

Putting both inequalities (32) and (36) together, we get an estimate in the space B (recall (28)), namely,

$$\big\|\omega(t)\big\|_B \leq C \|\omega_0\|_B \left\{ N + 2^{-N\epsilon} \exp\left(C \int_0^t \big\|\nabla u(s)\big\|_{L^\infty} \mathrm{d}s\right)\right\}, \tag{37}$$

and chosing N such that $N\epsilon \log(2) \approx 1 + C\int_0^t \|\nabla u\|$, we finally have the linear estimate

$$\big\|\omega(t)\big\|_B \leq C \|\omega_0\|_B \left(1 + \int_0^t \big\|\nabla u(s)\big\|_{L^\infty} \mathrm{d}s\right). \tag{38}$$

This type of (linear) estimate first originates with the work of M. Vishik ([6], 1998), and was later improved by T. Hmidi and S. Keraani with an application to the 2D Euler problem ([5], 2005).

4.5 Estimate for the velocity

Now that we have estimate (38) at our disposal, all that remains to do is to find an estimate for the velocity field using $\|\omega\|_B$. Remeber that we had expressed u as a function of the vorticity in equation (22):

$$\nabla u = \nabla \nabla^\perp (-\Delta)^{-1} \omega. \tag{39}$$

Since we had bounds on ω involving the Banach space B, it seems natural to use the Littlewood-Paley decomposition:

$$\big\|\nabla u\big\|_{L^\infty} \leq \big\|\nabla \Delta_{-1} u\big\|_{L^\infty} + \sum_{j\neq -1} \big\|\Delta_j \nabla u\big\|_{L^\infty}. \tag{40}$$

[7] Remember that the philosophy of the Littlewood-Paley decomposition is that the factor $2^{j\epsilon}$ is associated with a differentiation of order ϵ. The exponential factor appears for more or less the same reasons than in the case of the order one estimate.

Recall that the Fourier transform is a continuous operator in the $L^1 \longrightarrow L^\infty$ topology. Therefore, the low frequency term is bounded by

$$\left\|\nabla\Delta_{-1}u\right\|_{L^\infty} = \left\|\nabla\nabla^\perp(-\Delta)^{-1}\Delta_{-1}\omega\right\|_{L^\infty} \leq \int \varphi_{-1}(\xi)\left|m(\xi)\,\widehat{\omega}(\xi)\right|\mathrm{d}\xi, \tag{41}$$

where $m(\xi)$ is the symbol of the Fourier multiplier $\nabla\nabla^\perp(-\Delta)^{-1}$ and is a bounded homogeneous function of order zero. Remebering that φ_{-1} is supported in a ball, the Cauchy-Schwarz inequality provides

$$\left\|\nabla\Delta_{-1}u\right\|_{L^\infty} \leq C\|u\|_{L^2}, \tag{42}$$

and this last term is bounded because the L^2 norm of the velocity field is constant.

On the other hand, by using the Bernstein inequality from Lemma 4.2, we can bound the high frequency terms: because the operator $\nabla\nabla^\perp(-\Delta)^{-1}$ is (broadly speaking) a (pseudo)differential operator of order zero,

$$\sum_{j\neq-1}\left\|\Delta_j\nabla\nabla^\perp(-\Delta)^{-1}\omega\right\|_{L^\infty} \leq C\sum_{j\neq-1}\|\Delta_j\omega\|_{L^\infty} \leq C\|\omega\|_B. \tag{43}$$

Putting together the low and high frequency estimates, we get

$$\|\nabla u\|_{L^\infty} \leq C\Big(\|u_0\|_{L^2} + \|\omega\|_B\Big). \tag{44}$$

This allows us to end our computation: using this in the linear estimate (38),

$$\left\|\omega(t)\right\|_B \leq C\|\omega_0\|_B\left(1 + t\|u_0\|_{L^2} + \int_0^t \left\|\omega(s)\right\|_B\mathrm{d}s\right). \tag{45}$$

Grönwall's lemma finally gives

$$\left\|\omega(t)\right\|_B \leq C\big(1 + t\,\|u_0\|_{L^2}\big)\exp\big(Ct\,\|\omega_0\|_B\big) \tag{46}$$

which is sufficient for our purposes.

5 Further remarks and results

As we end our discussion, several remarks are in order. We see that the a crucial feature of our proof is that estimate (38) is *linear* with respect to ∇u. That would not be the case if the vorticity equation involved an extra righthand term. This makes the case of 2D Euler equations very special: whenever the problem is perturbed in any way, the linearity in (38) is lost and we can no longer get *global* estimates.

For example, in three dimensions of space, when computing the curl of the momentum equation, we get again an equation solved by the vorticity, but with an extra so-called *stretching* term: if $\omega = \nabla \times u$ is the vorticity, then

$$\partial_t\omega + (u\cdot\nabla)\,\omega = \omega.\nabla u. \tag{47}$$

All we can hope for with this method, is estimates on a finite time intervall, which yield existence of solutions on that same time intervall. As a matter of fact, no global existence results are known for 3D Euler equations with general initial data. We refer to [1] (Chapter 7) for a detailed discussion.

Another example of problem where this approach fails is the incompressible density-dependent Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = 0 \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \operatorname{div}(u) = 0, \end{cases} \tag{48}$$

where ρ is the fluid density. In the very same way, no global existence results are known for these equations (even when $d = 2!$), because the vorticity equation has an additional righthand term. In the 2D case, these are

$$\partial_t \omega + u \cdot \nabla \omega = -\nabla \frac{1}{\rho} \wedge \nabla p. \tag{49}$$

However, if the density is close to being constant $\rho \approx 1$, then the equations are very near to the usual Euler system (1). In that case, we can expect better estimates. This is indeed the case: solutions with $\omega_0 \in B$ exist, and their lifespan can be shown to be arbitrarily large as ρ gets closer to the constant density 1. We refer to [3] for more on that topic.

References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier analysis and nonlinear partial differential equations*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011.

[2] A. L. Bertozzi and A. J. Majda: *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, Cambridge, UK, 2002.

[3] R. Danchin and F. Fanelli: *The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces*. J. Math. Pures Appl. (9) 96 (2011), n. 3, pp. 253-278.

[4] L. C. Evans: *Partial Differential Equations*. 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, USA, 2010.

[5] T. Hmidi and S. Keraani: *Existence globale pour le système d'Euler incompressible 2-D dans* $B^1_{\infty,1}$: C. R. Math. Acad. Sci. Paris. 341 (2005), n. 11, pp. 655-658.

[6] M. Vishik: *Hydrodynamics in Besov spaces*: Arch. Ration. Mech. Anal. 145 (1998), n. 3, pp. 197-214.