Higher Thurston Theory

Or the study of representations of surface groups in $PSL(n,\mathbb{R})$
as a a generalisation of Teichmüller-Thurston theory

François LABOURIE
Université Paris-Sud (Orsay)

How to make fun
with cross ratios ...

Disclaimer

Most of the statements and definitions below are not precise enough and should not be taken seriously !
The slides are meant to be complemented by a talk
Preprints, with accurate statements and bibliography can be downloaded from
HTTP://MATH.U-PSUD.FR/ LABOURIE/PREPRINTS/INDEX.HTML

Riemann Uniformisation Theorem

- Let S be a compact, connected surface of genus ≥ 2
- Let $\pi_1(S)$ be its fundamental group, also called *surface group*.

Riemann Uniformisation Theorem

- Let S be a compact, connected surface of genus ≥ 2
- Let $\pi_1(S)$ be its fundamental group, also called *surface group*.

RIEMANN UNIFORMISATION THEOREM:

$$\text{complex structure } J \text{ on } S \rightsquigarrow$$
$$\text{a representation } \rho : \pi_1(S) \to PSL(2, \mathbb{R})$$
$$\text{s.t. } (S, J) \text{ is conformal to } \mathbb{H}^2/\rho(\pi_1(S)) \quad .$$

Riemann Uniformisation Theorem

- Let S be a compact, connected surface of genus ≥ 2
- Let $\pi_1(S)$ be its fundamental group, also called *surface group*.

RIEMANN UNIFORMISATION THEOREM:

$$\text{complex structure } J \text{ on } S \rightsquigarrow$$

$$\text{a representation } \rho : \pi_1(S) \to PSL(2,\mathbb{R})$$
$$\text{s.t. } (S, J) \text{ is conformal to } \mathbb{H}^2/\rho(\pi_1(S)) \quad .$$

Such a representation ρ is said to be *Fuchsian* , it is the *monodromy of a hyperbolic structure.* Fuchsian representations fill up two isomorphic connected components of the space of representations [Goldman].

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

- The space of complex structures on S.

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

- The space of complex structures on S.
- a connected component of the space of representations $\mathrm{Rep}(\pi_1(S), PSL(2,\mathbb{R}))$.

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

- The space of complex structures on S.
- a connected component of the space of representations $\mathrm{Rep}(\pi_1(S), PSL(2,\mathbb{R}))$.
- The space of hyperbolic metrics on S.

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

- The space of complex structures on S.
- a connected component of the space of representations $\mathrm{Rep}(\pi_1(S), PSL(2,\mathbb{R}))$.
- The space of hyperbolic metrics on S.

The corresponding topological space $\mathcal{T}(S)$ is called *Teichmüller space.*

Teichmüller Spaces

Using Riemann's Theorem, we can identify three different moduli spaces

- The space of complex structures on S.
- a connected component of the space of representations $\mathrm{Rep}(\pi_1(S), PSL(2,\mathbb{R}))$.
- The space of hyperbolic metrics on S.

The corresponding topological space $\mathcal{T}(S)$ is called *Teichmüller space.*

We will first explain another elementary point of view on Teichmüller space.

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

This space has the following properties

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

This space has the following properties

- it is homeomorphic to the circle

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

This space has the following properties

- it is homeomorphic to the circle
- Every non trivial element γ of $\pi_1(S)$ has exactly two fixed points on $\partial_\infty \pi_1(S)$.
  - one attractive fixed point γ^+,
  - one repulsive fixed point γ^-.

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

This space has the following properties

- it is homeomorphic to the circle
- Every non trivial element γ of $\pi_1(S)$ has exactly two fixed points on $\partial_\infty \pi_1(S)$.
  - one attractive fixed point γ^+,
  - one repulsive fixed point γ^-.
- every orbit is dense.

The boundary at infinity of a surface group

To a surface group $\pi_1(S)$, we can associate a topological space , the *boundary at infinity* denoted $\partial_\infty \pi_1(S)$ on which $\pi_1(S)$ acts. It is is defined as the "horizon" of the group $\pi_1(S)$ viewed as a geometric object.

This space has the following properties

- it is homeomorphic to the circle
- Every non trivial element γ of $\pi_1(S)$ has exactly two fixed points on $\partial_\infty \pi_1(S)$.
  - one attractive fixed point γ^+,
  - one repulsive fixed point γ^-.
- every orbit is dense.

These three properties characterise the boundary at infinity [Gabai].

Geometric realisation of $\partial_\infty \pi_1(S)$

Let us choose a hyperbolic metric on S. The universal cover of S is identified with the hyperbolic plane $\mathbb{H}^2$.

Geometric realisation of $\partial_\infty \pi_1(S)$

Let us choose a hyperbolic metric on S. The universal cover of S is identified with the hyperbolic plane $\mathbb{H}^2$.

Then $\partial_\infty \pi_1(S)$ is identified with the boundary of the disk in the Poincaré disk model : $\partial_\infty \pi_1(S) = \partial_\infty \mathbb{H}^2 = \mathbb{RP}^1$.

Geometric realisation of $\partial_\infty \pi_1(S)$

Let us choose a hyperbolic metric on S. The universal cover of S is identified with the hyperbolic plane $\mathbb{H}^2$.

Then $\partial_\infty \pi_1(S)$ is identified with the boundary of the disk in the Poincaré disk model : $\partial_\infty \pi_1(S) = \partial_\infty \mathbb{H}^2 = \mathbb{RP}^1$.

hyperbolic structure on $S \rightsquigarrow$ identification $\partial_\infty \pi_1(S) \leftrightarrow \mathbb{RP}^1$

Geometric realisation of $\partial_\infty \pi_1(S)$

Let us choose a hyperbolic metric on S. The universal cover of S is identified with the hyperbolic plane $\mathbb{H}^2$.

Then $\partial_\infty \pi_1(S)$ is identified with the boundary of the disk in the Poincaré disk model : $\partial_\infty \pi_1(S) = \partial_\infty \mathbb{H}^2 = \mathbb{RP}^1$.

hyperbolic structure on $S \rightsquigarrow$ identification $\partial_\infty \pi_1(S) \leftrightarrow \mathbb{RP}^1$

Teichmüller space $\mathcal{T}(S) =$
moduli space of $\pi_1(S)$ invariant projective structures on $\partial_\infty \pi_1(S)$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \rightarrow \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

$$\begin{aligned} b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t) \\ b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t) \\ b(x,y,z,t) &= b(z,t,x,y) \\ b(x,y,z,t) &= 0 \quad \Leftrightarrow x=y \ \text{ or } z=t \\ b(x,y,z,t) &= 1 \quad \Leftrightarrow x=z \ \text{ or } y=t \end{aligned}$$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

$$\begin{aligned}
b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t) \\
b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t) \\
b(x,y,z,t) &= b(z,t,x,y) \\
b(x,y,z,t) &= 0 \quad \Leftrightarrow x = y \ \text{ or } z = t \ \text{(Normalisation)} \\
b(x,y,z,t) &= 1 \quad \Leftrightarrow x = z \ \text{ or } y = t \ \text{(Normalisation)}
\end{aligned}$$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

$$\begin{aligned}
b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t) \\
b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t) \\
b(x,y,z,t) &= b(z,t,x,y) \text{ (Symmetry)} \\
b(x,y,z,t) &= 0 \quad \Leftrightarrow x = y \ \text{ or } z = t \text{ (Normalisation)} \\
b(x,y,z,t) &= 1 \quad \Leftrightarrow x = z \ \text{ or } y = t \text{ (Normalisation)}
\end{aligned}$$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

$$\begin{aligned}
b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t) \\
b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t) \text{ (Symmetric rule)} \\
b(x,y,z,t) &= b(z,t,x,y) \text{ (Symmetry)} \\
b(x,y,z,t) &= 0 \quad \Leftrightarrow x = y \ \text{ or } z = t \text{ (Normalisation)} \\
b(x,y,z,t) &= 1 \quad \Leftrightarrow x = z \ \text{ or } y = t \text{ (Normalisation)}
\end{aligned}$$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules

$$\begin{aligned}
b(x,y,z,t) &= b(x,y,z,w)b(x,w,z,t) \text{ (Cocycle identity)}\\
b(x,y,z,t) &= b(x,y,w,t)b(w,y,z,t) \text{ (Symmetric rule)}\\
b(x,y,z,t) &= b(z,t,x,y) \text{ (Symmetry)}\\
b(x,y,z,t) &= 0 \quad \Leftrightarrow x = y \quad \text{or } z = t \text{ (Normalisation)}\\
b(x,y,z,t) &= 1 \quad \Leftrightarrow x = z \quad \text{or } y = t \text{ (Normalisation)}
\end{aligned}$$

Cross ratio and periods

- A *cross ratio* on $\partial_\infty \pi_1(S)$ is a Hölder function

$$b : \partial_\infty \pi_1(S)^4 \setminus \Delta \to \mathbb{R}^*,$$

invariant under the action of $\pi_1(S)$ satisfying some rules :

  – Cocycle identity and symmetric cocycle identity

  – Symmetry and normalisation

- Let $\gamma \in \pi_1(S)$. The *period* of γ with respect to a cross ratio b is

$$l_b(\gamma) = \log |b(\gamma^+, y, \gamma^-, \gamma y)|.$$

Where γ^+ (resp. γ^-) is the attracting (resp. repelling) fixed point of γ on $\partial_\infty \pi_1(S)$, and y any element in $\partial_\infty \pi_1(S)$.

Classical Example

- The *classical* cross ratio on $\mathbb{RP}^1$, satisfies furthermore the functional relation.

$$F(2) : b(x, y, z, t) = 1 - b(t, y, z, x).$$

Moreover, every cross ratio satisfying $F(2)$ is classical.

Classical Example

- The *classical* cross ratio on $\mathbb{RP}^1$, satisfies furthermore the functional relation.

$$F(2) : b(x, y, z, t) = 1 - b(t, y, z, x).$$

Moreover, every cross ratio satisfying $F(2)$ is classical.

- In particular

$$\rho \in \mathcal{T}(S) \rightsquigarrow \text{ cross ratio on } \partial_\infty \pi_1(S) \simeq \mathbb{RP}^1.$$

Classical Example

- The *classical* cross ratio on $\mathbb{RP}^1$, satisfies furthermore the functional relation.

$$F(2) : b(x, y, z, t) = 1 - b(t, y, z, x).$$

Moreover, every cross ratio satisfying $F(2)$ is classical.

- In particular

$$\rho \in \mathcal{T}(S) \rightsquigarrow \text{ cross ratio on } \partial_\infty \pi_1(S) \simeq \mathbb{RP}^1.$$

Teichmüller space $\mathcal{T}(S) =$
space of cross ratios on $\partial_\infty \pi_1(S)$ satisfying $F(2)$.
periods $\leftrightsquigarrow$ lengths of closed geodesics

Less Classical Examples

- if ξ and ξ^* are curves from S^1 to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively,

$$b(x,y,z,t) = \frac{\langle\hat{\xi}(x), \hat{\xi}^*(y)\rangle\langle\langle\hat{\xi}(z), \hat{\xi}^*(t)\rangle}{\langle\hat{\xi}(z), \hat{\xi}^*(y)\rangle\langle\hat{\xi}(x), \hat{\xi}^*(t)\rangle}.$$

(Generalisation to flag manifolds)

Less Classical Examples

- if ξ and ξ^* are curves from S^1 to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively,

$$b(x,y,z,t) = \frac{\langle\hat{\xi}(x),\hat{\xi}^*(y)\rangle\langle\langle\hat{\xi}(z),\hat{\xi}^*(t)\rangle}{\langle\hat{\xi}(z),\hat{\xi}^*(y)\rangle\langle\hat{\xi}(x),\hat{\xi}^*(t)\rangle}.$$

  (Generalisation to flag manifolds)

- *Geodesic flows* of negatively curved metrics on $S \rightsquigarrow$ crossratios Periods = length of closed geodesics. (In general, Anosov flows $\leftrightsquigarrow$ Cross ratios) [Otal, Hamenstädt, Ledrappier]

Less Classical Examples

- if ξ and ξ^* are curves from S^1 to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively,

$$b(x,y,z,t) = \frac{\langle\hat{\xi}(x),\hat{\xi}^*(y)\rangle\langle\langle\hat{\xi}(z),\hat{\xi}^*(t)\rangle}{\langle\hat{\xi}(z),\hat{\xi}^*(y)\rangle\langle\hat{\xi}(x),\hat{\xi}^*(t)\rangle}.$$

  (Generalisation to flag manifolds)

- *Geodesic flows* of negatively curved metrics on $S \rightsquigarrow$ crossratios Periods = length of closed geodesics. (In general, Anosov flows $\leftrightsquigarrow$ Cross ratios) [Otal, Hamenstädt, Ledrappier]

We will show that Teichmüller theory extends as a higher Teichmüller theory (or higher Thurston theory) which is a dictionary between cross ratios, representations of surface groups and complex analysis.

Representations of $\pi_1(S)$ in $PSL(n, \mathbb{R})$

Representations of $\pi_1(S)$ in $PSL(n, \mathbb{R})$

- *Fuchsian representations* in $PSL(n, \mathbb{R})$ By definition these are compositions

$$\pi_1(S) \stackrel{\text{Fuchsian}}{\longrightarrow} PSL(2, \mathbb{R}) \stackrel{\text{irreducible}}{\longrightarrow} PSL(n, \mathbb{R}).$$

Representations of $\pi_1(S)$ in $PSL(n,\mathbb{R})$

- *Fuchsian representations* in $PSL(n,\mathbb{R})$ By definition these are compositions

$$\pi_1(S) \stackrel{\text{Fuchsian}}{\longrightarrow} PSL(2,\mathbb{R}) \stackrel{\text{irreducible}}{\longrightarrow} PSL(n,\mathbb{R}).$$

- *Hitchin representations* in $PSL(n,\mathbb{R})$: a representation which can be *deformed* to a Fuchsian representation.

Representations of $\pi_1(S)$ in $PSL(n, \mathbb{R})$

- *Fuchsian representations* in $PSL(n, \mathbb{R})$ By definition these are compositions

$$\pi_1(S) \overset{\text{Fuchsian}}{\longrightarrow} PSL(2, \mathbb{R}) \overset{\text{irreducible}}{\longrightarrow} PSL(n, \mathbb{R}).$$

- *Hitchin representations* in $PSL(n, \mathbb{R})$: a representation which can be *deformed* to a Fuchsian representation.

- *Hitchin component* $\mathcal{H}(n, S)$ is (one of) the connected component(s) of

$$\{\text{Hitchin representations}\}/PSL(n, \mathbb{R}).$$

A Theorem by Hitchin

Observe first that $\mathcal{H}(2, S) = \mathcal{T}(S)$.

A Theorem by Hitchin

Observe first that $\mathcal{H}(2, S) = \mathcal{T}(S)$.

Theorem [Hitchin-90] *The Hitchin component* $\mathcal{H}(n, S)$ *is homeomorphic to a ball of dimension* $-\chi(S)(n^2 - 1)$*.*

The proof uses harmonic mappings.

A Theorem by Hitchin

Observe first that $\mathcal{H}(2, S) = \mathcal{T}(S)$.

Theorem [Hitchin-90] *The Hitchin component* $\mathcal{H}(n, S)$ *is homeomorphic to a ball of dimension* $-\chi(S)(n^2 - 1)$.

The proof uses harmonic mappings.

- What can be said individually about Hitchin's representations? are they discrete?
- What about the mapping class group $\mathrm{Out}(\pi_1(S))$ action on $\mathcal{H}(n, S)$?

A Theorem by Hitchin

Observe first that $\mathcal{H}(2, S) = \mathcal{T}(S)$.

Theorem [Hitchin-90] *The Hitchin component* $\mathcal{H}(n, S)$ *is homeomorphic to a ball of dimension* $-\chi(S)(n^2 - 1)$.

The proof uses harmonic mappings.

- What can be said individually about Hitchin's representations? are they discrete?
- What about the mapping class group $\mathrm{Out}(\pi_1(S))$ action on $\mathcal{H}(n, S)$?
- Are Hitchin representations symmetries of geometric objects?

Theorem [L.]

- *Hitchin component* $\mathcal{H}(n, S)$ = *space of cross ratios on* $\partial_\infty \pi_1(S)$ *satisfying a rational functional relation* $F(n)$.

Theorem [L.]

- *Hitchin component* $\mathcal{H}(n, S)$ = *space of cross ratios on* $\partial_\infty \pi_1(S)$ *satisfying a rational functional relation* $F(n)$.

- *In this correspondence,*

$$\forall \gamma \in \pi_1(S), \quad l_b(\gamma) = \log \frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma)}.$$

Theorem [L.]

- *Hitchin component* $\mathcal{H}(n,S)$ = *space of cross ratios on* $\partial_\infty \pi_1(S)$ *satisfying a rational functional relation* $F(n)$.

- *In this correspondence,*

$$\forall \gamma \in \pi_1(S), \quad l_b(\gamma) = \log \frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma)}.$$

- *Every Hitchin representation is discrete, faithful, the image of every element in* $\pi_1(S)$ *is real split.*

Theorem [L.]

- *Hitchin component* $\mathcal{H}(n, S)$ = *space of cross ratios on* $\partial_\infty \pi_1(S)$ *satisfying a rational functional relation* $F(n)$.

- *In this correspondence,*

$$\forall \gamma \in \pi_1(S), \quad l_b(\gamma) = \log \frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma)}.$$

- *Every Hitchin representation is discrete, faithful, the image of every element in* $\pi_1(S)$ *is real split.*

- *The mapping class group acts properly on* $\mathcal{H}(n)$.

The functional relation $F(n)$

$F(n)$ involves quotient of determinants of matrices whose coefficients are values of the cross ratio.

The functional relation $F(n)$

$F(n)$ involves quotient of determinants of matrices whose coefficients are values of the cross ratio.If $\{\ldots, e_i, \ldots, u_i \ldots\}$ are points in $\partial_\infty \pi_1(S)$, we define

$$\chi^n(e_0, \ldots, e_n; u_0, \ldots, u_n) = \det(b(e_i, u_j, e_0, u_0)).$$

The functional relation $F(n)$

$F(n)$ involves quotient of determinants of matrices whose coefficients are values of the cross ratio.If $\{\ldots, e_i, \ldots, u_i \ldots\}$ are points in $\partial_\infty \pi_1(S)$, we define

$$\chi^n(e_0, \ldots, e_n; u_0, \ldots, u_n) = \det(b(e_i, u_j, e_0, u_0)).$$

Then $F(n)$ is

$$\begin{aligned} \forall p \leq n,\ \chi^p &\neq 0, \\ \forall p > n,\ \chi^p &= 0. \end{aligned}$$

In particular,

$$\chi^{n+1} = 0.$$

Hyperconvex curves

A continuous curve ξ from S^1 to $\mathbb{RP}^{n-1}$ is *hyperconvex* if for any distinct points $(x_1, \ldots, x_n)$ in S^1 we have

$$\xi(x_1) \oplus \ldots \oplus \xi(x_n) = \mathbb{R}^n.$$

Hyperconvex curves

A continuous curve ξ from S^1 to $\mathbb{RP}^{n-1}$ is *hyperconvex* if for any distinct points $(x_1, \ldots, x_n)$ in S^1 we have

$$\xi(x_1) \oplus \ldots \oplus \xi(x_n) = \mathbb{R}^n.$$

- for $n = 2$ hyperconvex means injective,
- for $n = 3$ hyperconvex means convex,
- The Veronese embedding is hyperconvex.
- By Cauchy-Crofton formula, hyperconvex curves are rectifiable with universally bounded length :

$$\text{length(c)} = \int_{\text{hyperplanes P}} \underbrace{\sharp(c \cap P)}_{\leq n-1} dP.$$

Theorem [L.] *If* ρ *is a Hitchin representation, then there exists a (unique)* ρ*-equivariant hyperconvex curve* ξ*, the* limit curve*, from* $\partial_\infty \pi_1(S)$ *to* $\mathbb{RP}^{n-1}$*. Furthermore, these (generally only* C^1*) curves admits continuous osculating flags in general position*

Theorem [L.] *If ρ is a Hitchin representation, then there exists a (unique) ρ-equivariant hyperconvex curve ξ, the* limit curve*, from $\partial_\infty \pi_1(S)$ to $\mathbb{RP}^{n-1}$. Furthermore, these (generally only C^1) curves admits continuous osculating flags in general position*

- for Fuchsian representations, we get the Veronese embedding,
- Hitchin representations are symmetries of hyperconvex curves.

Theorem [L.] *If ρ is a Hitchin representation, then there exists a (unique) ρ-equivariant hyperconvex curve ξ, the* limit curve*, from $\partial_\infty \pi_1(S)$ to $\mathbb{RP}^{n-1}$. Furthermore, these (generally only C^1) curves admits continuous osculating flags in general position*

- for Fuchsian representations, we get the Veronese embedding,
- Hitchin representations are symmetries of hyperconvex curves.
- This leads to cross ratios.

Theorem [L.] *If ρ is a Hitchin representation, then there exists a (unique) ρ-equivariant hyperconvex curve ξ, the* limit curve*, from $\partial_\infty \pi_1(S)$ to $\mathbb{RP}^{n-1}$. Furthermore, these (generally only C^1) curves admits continuous osculating flags in general position*

- for Fuchsian representations, we get the Veronese embedding,
- Hitchin representations are symmetries of hyperconvex curves.
- This leads to cross ratios.

Using a result by Guichard, we have

Hitchin component $\mathcal{H}(n, S) =$ moduli space of equivariant hyperconvex curves in $\mathbb{RP}^{n-1}$

Universal constructions

Universal constructions

- *Universal Hitchin component* (when genus of S goes to infinity)
  $= \{\text{hyperconvex curves }\}$

Universal constructions

- *Universal Hitchin component* (when genus of S goes to infinity) $= \{\text{hyperconvex curves}\}$
- *Universal Hitchin component* (when n goes to infinity) =

$$\mathrm{Rep}^*(\pi_1(S), C^{1,h}(S^1) \rtimes \mathit{Diff}^h(S^1)) = \mathcal{H}(\infty, S)$$

Universal constructions

- *Universal Hitchin component* (when genus of S goes to infinity) $= \{$hyperconvex curves $\}$
- *Universal Hitchin component* (when n goes to infinity) $=$

$$\mathrm{Rep}^*(\pi_1(S), C^{1,h}(S^1) \rtimes \mathit{Diff}^h(S^1)) = \mathcal{H}(\infty, S)$$

We have the two inclusions

$$\begin{aligned} \forall n, \quad \mathcal{H}(n,S) \quad &\subset \quad \mathcal{H}(\infty, S) \\ \{\text{ negatively curved metrics on } S\} \quad &\subset \quad \mathcal{H}(\infty, S). \end{aligned}$$

A conjecture about uniformisation

A conjecture about uniformisation

- Let's fix a complex structure J on S, N. Hitchin constructed a homeomorphism (actually a section of Hitchin's fibration)

$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)$$

A conjecture about uniformisation

- Let's fix a complex structure J on S, N. Hitchin constructed a homeomorphism (actually a section of Hitchin's fibration)

$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)$$

- We want to get rid of the choice of J.

A conjecture about uniformisation

- Let's fix a complex structure J on S, N. Hitchin constructed a homeomorphism (actually a section of Hitchin's fibration)

$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)$$

- We want to get rid of the choice of J. Let $\mathcal{T}(S)$ be Teichmüller space. Let $E \to \mathcal{T}(S)$ be the vector bundle with fibre

$$E_J = H^0(K_J^3) \oplus \ldots \oplus H^0(K_J^n).$$

A conjecture about uniformisation

- Let's fix a complex structure J on S, N. Hitchin constructed a homeomorphism (actually a section of Hitchin's fibration)

$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)$$

- We want to get rid of the choice of J. Let $\mathcal{T}(S)$ be Teichmüller space. Let $E \to \mathcal{T}(S)$ be the vector bundle with fibre

$$E_J = H^0(K_J^3) \oplus \ldots \oplus H^0(K_J^n).$$

Let

$$\Phi : \begin{cases} E & \mapsto \mathcal{H}(n) \\ (J, \omega) & \to \phi_J(0, \omega) \end{cases}$$

A conjecture about uniformisation

- Let's fix a complex structure J on S, N. Hitchin constructed a homeomorphism (actually a section of Hitchin's fibration)

$$\phi_J : H^0(K_J^2) \oplus \ldots \oplus H^0(K_J^n) \to \mathcal{H}(n)$$

- We want to get rid of the choice of J. Let $\mathcal{T}(S)$ be Teichmüller space. Let $E \to \mathcal{T}(S)$ be the vector bundle with fibre

$$E_J = H^0(K_J^3) \oplus \ldots \oplus H^0(K_J^n).$$

Let

$$\Phi : \begin{cases} E & \mapsto \quad \mathcal{H}(n) \\ (J, \omega) & \to \quad \phi_J(0, \omega) \end{cases}$$

Notice that Φ is equivariant under the action of the Mapping Class Group $\mathcal{M}(S) = Out(\Gamma)$.

Conjecture *The map* Φ *is a homeomorphism*

Conjecture *The map* Φ *is a homeomorphism*

- For $n = 2$, this is Riemann's Theorem.
- This conjecture has a translation as the stability of a minimal surface in a symmetric space.
- This would imply $\mathcal{H}(n)/\mathcal{M}(S)$ is naturally a complex manifold.

Conjecture *The map* Φ *is a homeomorphism*

- For $n = 2$, this is Riemann's Theorem.
- This conjecture has a translation as the stability of a minimal surface in a symmetric space.
- This would imply $\mathcal{H}(n)/\mathcal{M}(S)$ is naturally a complex manifold.

Theorem [L.] Φ *is surjective : this follows from the existence for every Hitchin representation* ρ *of a minimal surface in*

$$\rho(\pi_1(S))\backslash SL(n, \mathbb{R})/SO(n, \mathbb{R})$$

Moreover, for $n = 3$*,* Φ *is a homeomorphism.*

Teichmüller-Thurston dictionnary

Teichmüller-Thurston theory $\rightsquigarrow$ a dictionnary between complex analysis, algebra (representations of surface groups), geometry (hyperbolic metrics), dynamics, combinatorics , integrable systems (Virasoro algebra).

Teichmüller-Thurston dictionnary

Teichmüller-Thurston theory $\rightsquigarrow$ a dictionnary between complex analysis, algebra (representations of surface groups), geometry (hyperbolic metrics), dynamics, combinatorics (Penner's cell decomposition), integrable systems (Virasoro algebra).

What can be said about representations of $\pi_1(S)$ *in* $PSL(n, \mathbb{R})$ *?*

Other elements of the dictionnary

Other elements of the dictionnary

- [COMBINATORICS] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups.

Other elements of the dictionnary

- [Combinatorics] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups.

- Mc Shane's identity about length of closed geodesics $\rightsquigarrow$ higher identities [L-, Mc Shane]

Other elements of the dictionnary

- [COMBINATORICS] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups

- Mc Shane's identity about length of closed geodesics $\rightsquigarrow$ higher identities [L-, Mc Shane]

- Dynamical properties of the geodesic flow $\rightsquigarrow$ Anosov flow associated to Hitchin representation [L]

Other elements of the dictionnary

- [Combinatorics] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups
- Mc Shane's identity about length of closed geodesics $\rightsquigarrow$ higher identities [L-, Mc Shane]
- Dynamical properties of the geodesic flow $\rightsquigarrow$ Anosov flow associated to Hitchin representation [L]
- Properness of the energy functional [Schoen-Yau] remains true.

Other elements of the dictionnary

- [COMBINATORICS] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups

- Mc Shane's identity about length of closed geodesics $\rightsquigarrow$ higher identities [L-, Mc Shane]

- Dynamical properties of the geodesic flow $\rightsquigarrow$ Anosov flow associated to Hitchin representation [L]

- Properness of the energy functional [Schoen-Yau] remains true.

- Compactifications : A. Parreau

Other elements of the dictionnary

- [COMBINATORICS] Thurston shear coordinates on $\mathcal{T}(S)$ for surfaces with boundary
  $\rightsquigarrow$ Fock-Goncharov work : coordinates, quantisation, positive algebraic geometry interpretation, as well as extension to the open case and real-split groups.

- Mc Shane's identity about length of closed geodesics $\rightsquigarrow$ higher identities [L-, Mc Shane]

- Dynamical properties of the geodesic flow $\rightsquigarrow$ Anosov flow associated to Hitchin representation [L]

- Properness of the energy functional [Schoen-Yau] remains true.

- Compactifications : A. Parreau

- Integrable systems ? $W(n)$ algebras ? Intersection numbers ?