On geometry and combinatorics in representation theory

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title a little vague ... Aim:

explain what the *Saturation conjecture* is about (representation theoretic problem) and how it is related it to Horn's conjecture, and a few words about the more general setting (reductive algebraic groups)

Horn: eigenvalues sums of Hermitian matrices	$\leftrightarrow$	symplectic geometry
		$\updownarrow$
Saturation conjecture	$\leftrightarrow$	flag varieties
$\updownarrow$		
buildings		

Hermitian matrices and eigenvalues

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Reformulate:

Describe all triples (α, β, γ) of n-tuples of weakly decreasing real numbers

$$\alpha = (\alpha_1 \geq \ldots \geq \alpha_n), \quad \beta = (\beta_1 \geq \ldots \geq \beta_n),$$
$$\gamma = (\gamma_1 \geq \ldots \geq \gamma_n)$$

such there exist Hermitian matrices A, B, C having α, β, γ as spectra and $\mathbf{A} + \mathbf{B} = \mathbf{C}$.

Since $A + B = C \Rightarrow \text{Trace}(A) + \text{Trace}(B) = \text{Trace}(C)$, necessary condition on the triple (α, β, γ)

trace identity: $$\sum_i \gamma_i = \sum_j \alpha_j + \sum_k \beta_k$$

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$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad \text{for } 1 \leq i + j - 1 \leq n$$

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Aim: give necessary and sufficient conditions in the general case

Horn‘s conjecture

Horn's conjecture (1962): *Let α, β, γ be n-tuples of weakly decreasing real numbers. $\exists$ $A, B, C = A + B$ Hermitian matrices with spectra α, β, γ if and only if the trace identity holds*

$$\sum \gamma_k = \sum \alpha_i + \sum \beta_j. \tag{1}$$

and for all triples (I, J, K) in

$\mathcal{H}_r^n$ = *certain set of triples of subsets of* $\{1, \dots, n\}$, $|I| = |J| = |K| = r$

and all $r < n$ the following inequalities hold:

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \tag{2}$$

Horn's conjecture

Definition of $\mathcal{H}_r^n$:
elementary, but somewhat involved..., $I, J, K \subset \{1, \dots, n\}$

$$h_r^n := \left\{ (I,J,K) \mid \begin{array}{c} |I| = |J| = |K| = r \\ \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r+1)}{2} \end{array} \right\}$$

Set $\mathcal{H}_1^n = h_1^n$, write $I = \{i_1 \leq \ldots \leq i_r\}, \ldots$

$$\mathcal{H}_r^n := \left\{ (I,J,K) \in h_r^n \mid \begin{array}{c} \text{for all } p < r \text{ and all } (X,Y,Z) \in \mathcal{H}_p^r \\ \sum_{x \in X} i_x + \sum_{y \in Y} j_y \leq \sum_{z \in Z} k_z + \frac{p(p+1)}{2} \end{array} \right\}$$

Representations: some basic notation

Let us take as an example the symmetric group $\mathfrak{S}_4$ - several ways to think of the group,

1. group of bijections $\phi : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\}$, for example $(1, 2)$: $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$ and $4 \mapsto 4$.

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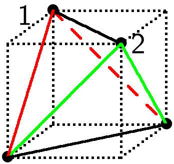
2. as the symmetry group of the tetrahedron and hence as a subgroup of $O_3(\mathbb{R})$.



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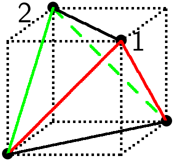

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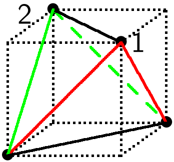
2. as the symmetry group of the tetrahedron and hence as a subgroup of $O_3(\mathbb{R})$. 

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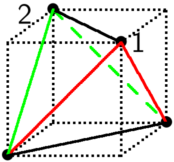

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Roughly: finite dimensional representation of a group = realization as a group of matrices

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2) tensor product: $\rho : G \longrightarrow GL(V_1 \otimes V_2)$,
$\rho(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$

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Example for a reducible representation:

$$\begin{array}{rcrcl} \mathbb{C}^n \otimes \mathbb{C}^n & = & \text{symmetric tensors} & \oplus & \text{skew symmetric tensors} \\ & = & S^2\mathbb{C}^n & \oplus & \Lambda^2\mathbb{C}^n \end{array}$$

Tensor product decomposition

n-tuples α, β weakly decreasing integers, tensor product

$$V(\alpha) \otimes V(\beta) = \bigoplus c_{\alpha,\beta}^{\gamma} V(\gamma)$$

Tensor product problem: describe all triples (α, β, γ) of n-tuples of weakly decreasing integers such that $c_{\alpha,\beta}^{\gamma} > 0$

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Saturation conjecture:
If $\exists N > 0$ *such that* $c_{N\alpha,N\beta}^{N\gamma} > 0$, *then* $c_{\alpha,\beta}^{\gamma} > 0$.

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Theorem (Klyachko, Knutson, Tao,...) *Horn's conjecture is true, Saturation conjecture holds, and if* α, β, γ *are n-tuples of weakly decreasing integers, then* $\exists$ *Hermitian matrices* $A, B, C = A + B$ *with eigenvalues* (α, β, γ) *if and only if* $c_{\alpha,\beta}^{\gamma} > 0$.

Symplectic manifold and moment map

Let (M, ω) be a manifold together with an anti-symmetric inner product on the tangent spaces.

Example: $\mathbb{R}^{2n}$, basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$,
$\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ for all $i, j = 1, \ldots, n$ and
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G connected Lie group, acting smoothly on (M, ω). A map $\mu : M \to \mathfrak{g}^*$ ($=$ dual of Lie $G = \mathfrak{g}$) is called a *moment map* if

1) *the map is equivariant, i.e.,* $\mu(g.m) = g\mu(m)g^{-1}$

2) for $k \in \mathfrak{g}$, the associated vector field is the symplectic gradient of $\langle k, \mu(\cdot)\rangle$.

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Language of group actions: $\mathcal{O}_\alpha$ = Hermitian matrices, spectrum α - orbit for unitary group $U_n(\mathbb{C}) = \{g \in GL_n(\mathbb{C}) \mid g\overline{g}^T = \mathrm{Id}\}$

$$\mathcal{O}_\alpha = \left\{ k \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{pmatrix} k^{-1} \mid k \in U_n(\mathbb{C}) \right\},$$

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Set $H =$ real space of Hermitian matrices. Consider the map

$$\begin{array}{rcl} \mu : \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma & \longrightarrow & H \\ (A, B, C) & \mapsto & A + B + C \end{array}$$

Matrix problem: describe all (α, β, γ) **such that** $0 \in \mathbf{Im}(\mu)$ (replace γ by $\gamma^* = (-\gamma_n, \ldots, -\gamma_1)$)

Moment map

Lie $U_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C})$ skew Hermitian matrices

dual space $\mathfrak{u}_n(\mathbb{C})^*$ can be identified with $H =$ Hermitian matrices by $H \ni X \mapsto \text{Trace}(-iX\cdot)$

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Consequence: $\mathcal{O}_\alpha$ has a unique structure of a symplectic manifold such that the inclusion $\mathcal{O}_\alpha \hookrightarrow H$ is the moment map.

Moment map

General theory implies in our case:

$$\begin{array}{rcl} \mu : \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma & \longrightarrow & H \\ (A, B, C) & \mapsto & A + B + C \end{array}$$

is a moment map for the action of the unitary group $U_n(\mathbb{C})$.

Geometric realization of representations

Geometric realization via flag varieties: associate a (partial) flag to $D(\alpha) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{pmatrix}$, let $\mathbb{C}_{\alpha_i}=$ eigenspace $\sim \alpha_i$:

$$\text{flag} \quad F : F_0 = \{0\} \subseteq \mathbb{C}^n_{\alpha_1} \subseteq \mathbb{C}^n_{\alpha_1} \oplus \mathbb{C}^n_{\alpha_2} \subseteq \ldots \subseteq \mathbb{C}^n.$$

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Borel-Weil: *(a version of...) $\mathcal{F}(\alpha) \subseteq \mathbb{P}(V(\alpha))$, and the homogeneous coordinate ring is, as $GL_n(\mathbb{C})$-representation, isomorphic to $\bigoplus_{k\geq 0} V(k\alpha)^*$.*

Back to the tensor product

Tensor product problem: Given n-tuples α, β, γ of weakly decreasing integers, what are the conditions such that $\big(V(\alpha) \otimes V(\beta) \otimes V(\gamma)\big)^{GL_n} \neq 0$, i.e., contains a GL_n-invariant line?

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Let α, β, γ be n-tuples of weakly decreasing integers, then

$$\mathcal{F}(\alpha) \times \mathcal{F}(\beta) \times \mathcal{F}(\gamma) \subseteq \mathbb{P}\big(V(\alpha) \otimes V(\beta) \otimes V(\gamma)\big)$$

One knows: smooth complex projective varieties have structure of a symplectic variety, and $\mathcal{F}(\alpha) \times \mathcal{F}(\beta) \times \mathcal{F}(\gamma)$ is isomorphic to $\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma$ as symplectic manifold

GIT-geometric invariant theory

The link between the two pictures:
GIT- (*very rough summary*) $X \subset \mathbb{P}(V)$ complex projective variety, action by a group G . Aim: parametrization (GIT-quotient) of G-orbits (classification problems,....).

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α, β, γ n-tuples of weakly decreasing integers

Kirwan-Ness *(application of ...) Consider the moment map* $\mu : \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma \to H$*. Then there is a natural identification between the orbit set*

$$\mu^{-1}(0)/U_n(\mathbb{C})$$

and the GIT-quotient

$$\mathcal{F}(\alpha) \times \mathcal{F}(\beta) \times \mathcal{F}(\gamma)//GL_n(\mathbb{C})$$
$$= Proj\left(\bigoplus_{k\geq 0}(V(k\alpha)^* \otimes V(k\beta)^* \otimes V(k\gamma)^*)^{GL_n(\mathbb{C})}\right).$$

Hermitian matrices and tensor product

It means for *n*-tuples of weakly decreasing integers α, β, γ :

$\exists\, \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$ with desired spectra $\alpha, \beta, \gamma \Leftrightarrow$

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is not the empty set.

This is a first step, but what about the inequalities and the saturation conjecture?

Some remarks about tensor product rules

For $GL_n(\mathbb{C})$ several tensor product rules: Steinberg formula, Littlewood-Richardson rule using Young tableaux, crystal bases methods, path model formula, none of it worked.

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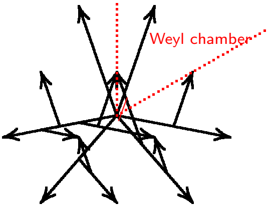
Different approach by Kapovich-Leeb-Millson using the theory of Euclidean buildings.

Path model and tensor product

Picture for SL_3, $\alpha = (3, 2, 0)$, associates a set of paths to $V(\alpha)$



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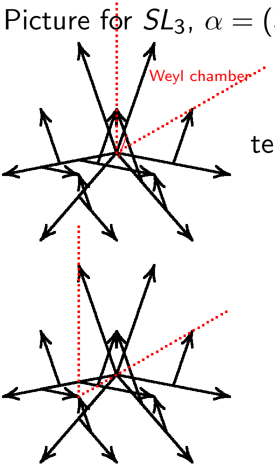

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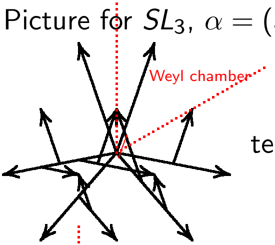
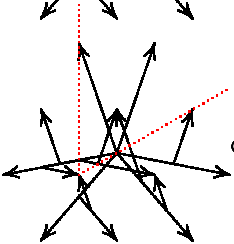
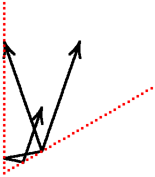


tensor product with $\mathbb{C}^3$ = shift by $\beta = (1,0,0)$



omit non-dominant paths



remaining paths =
irreducibles in tensor product decomposition.

Triangles in buildings

Problem: Why path for $V(N\gamma) \hookrightarrow V(N\alpha) \otimes V(N\beta)$ should be divisible by N to give $V(\gamma) \hookrightarrow V(\alpha) \otimes V(\beta)$?

Kapovich-Leeb-Millson consider

G split semisimple algebraic group over a locally compact field with discrete valuation, finite residue field, associate to this setting the Bruhat-Tits building X modeled on a Euclidean Coxeter complex.

Triangles in buildings

Example: apartment for SL_3

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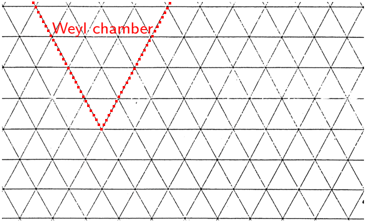

Triangles in buildings

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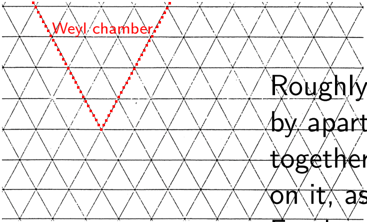
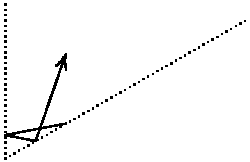


Roughly speaking, the building X is covered by apartments, i.e., real vector spaces together with an affine Weyl group acting on it, associated to the root system of G. Fundamental domain in an apartment $=$ alcove, any two alcoves in X are in one apartment.

Can retract X onto one apartment! and fold onto one Weyl chamber. Idea: lift the paths into the building

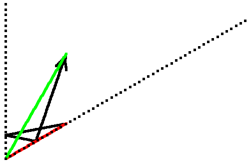
Triangles in buildings and saturation conjecture

Rough summary: start with $V(N\gamma) \hookrightarrow V(N\alpha) \otimes V(N\beta)$
start with path from model



Triangles in buildings and saturation conjecture

Rough summary: start with $V(N\gamma) \hookrightarrow V(N\alpha) \otimes V(N\beta)$
start with path from model, complete to *broken triangle*



Triangles in buildings and saturation conjecture

Rough summary: start with $V(N\gamma) \hookrightarrow V(N\alpha) \otimes V(N\beta)$
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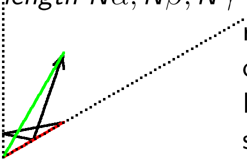
they lift / unfold it to a *geodesic* triangle in the building *of side length* $N\alpha, N\beta, N\gamma$ and show there exists (with some technical restrictions) also a geodesic triangle of side length α, β, γ.



Further, $\exists \ell$ (depends only on the root system) such that the image of a triangle of side of length $\ell\alpha, \ell\beta, \ell\gamma$ (again a broken triangle) corresponds to $V(\ell\gamma) \hookrightarrow V(\ell\alpha) \otimes V(\ell\beta)$.

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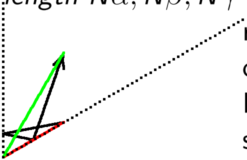

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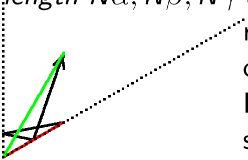

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Conjecture. G simply laced (SL_n, $Spin_{2n}$, E_6, E_7, E_8), then $\ell = 1$.

Graßmann variety

What about Horn's conjecture, where do the inequalities come from in this picture?

Fix a complete flag $F^\bullet : F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n$ i.e., a sequence of subspaces such that $\dim F_i = i$.

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$$\mathcal{F} : \{0\} \subset F_1 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \subset F_2 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle \subset F_3 = \mathbb{C}^3.$$

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Fix $r \in \{1, 2, \ldots, n-1\}$.
The **Graßmann variety** $G_{r,n}$ is the set of all subspaces U of $\mathbb{C}^n$ of dimension r. For example $G_{1,n} = \mathbb{P}^{n-1}$ is the projective space.

Schubert variety

Let $I \subset \{1, 2, ..., n\}$ be a subset of cardinality r, write $I = (i_1 < \ldots < i_r)$. The **Schubert variety** $X_I(\mathcal{F})$ is the subset

$$X_I(\mathcal{F}) = \{U \in G_{r,n} \mid \dim(U \cap F_{i_t}) \geq t \text{ for } 1 \leq t \leq r\}$$

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- let ω_I be its homology class (independent of $\mathcal{F}$)
- little technical twist: set $I^* = \{n + 1 - i \mid i \in I\}$,

often one uses for the cohomology classes as indexing $\lambda(I) = (i_r - r, \ldots, i_1 - 1)$ instead, let $\sigma_{\lambda(I)}$ be the cohomology class corresponding to ω_{I^*} via Poincaré duality.

Schubert variety

- the classes σ_α,

$$\alpha = (\alpha_1 \geq \ldots \geq \alpha_r), \quad \text{where } n - r \geq \alpha_1 \text{ and } \alpha_r \geq 0$$

form a basis of the integral cohomology ring of the **Graßmann variety** $G_{r,n}$.

One knows:

$$\sigma_\alpha \cdot \sigma_\beta = \sum d^\gamma_{\alpha,\beta} \sigma_\gamma$$

with $\sum \alpha_i + \sum \beta_j = \sum \gamma_k$.

Moreover, $c^\gamma_{\alpha,\beta} = d^\gamma_{\alpha,\beta}$, so Littlewood-Richardson coefficients = cohomology ring structure coefficients.

Rayleigh trace

Suppose A Hermitian matrix, eigenvalues $\alpha = (\alpha_1 \geq \ldots \geq \alpha_n)$, let $\{v_1, \ldots, v_n\}$ be an orthogonal basis of eigenvectors,

$$\mathcal{F}(A) := \{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \ldots \subset \mathbb{C}^n$$

complete flag.

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complete flag.Let $U \subset \mathbb{C}^n$ be an r-dimensional subspace, fix an orthonormal basis $\{u_1, \ldots, u_r\}$, set

$$R_A(U) := \sum_{j=1}^{r} (Au_j, u_j), \quad \text{then}$$

Proposition $\sum_{i \in I}^{r} \alpha_i = \min_{U \in X_I(\mathcal{F}(A))} R_A(U)$

Schubert calculus

Let A, B, C Hermitian matrices, spectra α, β, γ, $A + B = C$.

Claim: *Let I, J, K be subsets of $\{1, \ldots, n\}$ of cardinality r such that $c_{\lambda(I),\lambda(J)}^{\lambda(K)} \neq 0$. Then*

$$\sum_{k\in K} \gamma_k \leq \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j$$

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Idea: LR-coeficients $> 0 \Rightarrow$ product of cohomology classes $> 0 \Rightarrow$ intersection of the Schubert varieties
$X_{I^*}(\mathcal{F}(-A)) \cap X_{J^*}(\mathcal{F}(-B)) \cap X_K(\mathcal{F}(C)) \neq \emptyset$.

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Take a subspace U in the intersection, then

$$-\sum_{i \in I} \alpha_i - \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k \leq R_{-A}(U) + R_{-B}(U) + R_C(U) = 0.$$

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So we get: solutions of the matrix problem

$$\left\{ (\alpha,\beta,\gamma)\in\mathbb{R}^{3n} \mid \begin{array}{cc} \alpha_1\geq\ldots & \exists A,B,C \text{ Hermitian matrices} \\ \beta_1\geq\ldots & \text{spectra } \alpha,\beta,\gamma \\ \gamma_1\geq\ldots & A+B=C \end{array} \right\}$$

forms a polyhedral cone in $\mathbb{R}^{3n}$, the defining inequalities given by Horn's conjecture, trace identity,and entries weakly decreasing

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The integral points in the cone correspond to solutions of the tensor product problem

The set of inequalities given by Horn's conjecture is not minimal: Knutson, Tao and Woodward proved that it suffices to take those (I, J, K) such that $c_{\lambda(I),\lambda(J)}^{\lambda(K)} = 1$, and that this set is minimal.

What about the general case? (beyond $GL_n(\mathbb{C})$)

G connected semisimple complex algebraic group

Eigenvalue problem makes sense also in this case (going to compact subgroup...Weyl chamber)

also the tensor product problem

proof of the *weak equivalence* is the same

saturation conjecture still open in the simply laced case.

Another "lift" of the path model to the building using minimal galleries has been constructed by Gaussent and L, set of all "galleries" of the model is a projective variety (Bott-Samelson variety). Background: connection with intersection cohomology realization of representations (Mirkovich and Vilonen).

Some remarks

Here the "preimage" of one path naturally has the structure of a quasi-affine variety. Counting points (over finite fields) leads to formula for the structure constants of the spherical Hecke algebras (multiplication of Hall-Littlewood polynomials) (Schwer).

Same type of formulas also obtained by Kapovich-Leeb-Millson (counting number of geodesic triangles with fixed side lengths), what is the connection?

What about inequalities: work of Berenstein-Sjamaar, Leeb-Millson, using Schubert calculus, Belkale - Kumar were able to reduce the system of inequalities, but it is an open question whether this set is irredundant

Question about products of cohomology classes $\neq 0$, Belkale - Kumar got necessary conditions.

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