Third exercise sheet "Algebra II" winter term 2024/5.

Problem 1 (5 points)**.** *Let K be a field of characteristic $p > 0$, $x \in K \setminus K^p$ and $L = L(\sqrt[p]{x})$. Show that $\mathrm{Tr}_{L/K}(l) = 0$ and $\mathrm{N}_{L/K}(l) = l^p$ for all $l \in L$.*

Problem 2 (8 points)**.** *Let K be a field and L/K a finite field extension which is totally inseparable in the sense that every $l \in L \setminus K$ is inseparable over K. Show that $\mathrm{Tr}_{L/K}$ vanishes identically and that $\mathrm{N}_{L/K}(l) = l^{[L:K]}$ for all $l \in L$.*

Problem 3 (2 points)**.** *Let L/K be a finite field extension, $A \subseteq K$ a subring and let $l \in L$ be integral over K. Show that $\mathrm{Tr}_{L/K} l$ and $\mathrm{N}_{L/K} l$ are integral over K!*

Problem 4 (5 points)**.** *Let A be a PID, K the field of quotients of A, and $P \subseteq A \setminus \{0\}$ such that every multiplicative equivalence class of prime elements of K contains precisely one element of P. For $x \in K^\times$ let*

$$x = \varepsilon_x \prod_{p \in P} p^{v_p(x)}$$

be the unique decomposition of x into prime factors. Let $p \in P$, and put $v_p(0) = \infty$. Show that v_p satisfies

$$v_p(xy) = v_p(x) + v_p(y)$$
$$v_p(x+y) \geq \min\bigl(v_p(x), v_p(y)\bigr).$$

In particular, $A_{pA} = \bigl\{x \in K \bigm| v_p(x) \geq 0\bigr\}$ is a discrete valuation ring.

Solutions should be submitted to the tutor by e-mail before Friday November 1 24:00.