Nineth exercise sheet "Algebra II" winter term 2024/5.

Problem 1 (4 points). *In the situation of Problem 7 of the previous sheet, show that B/A is unramified at the prime ideals $\mathfrak{q} \in \mathrm{Spec} B$ not containing p.*

Problem 2 (6 points). *Let n be a natural number and $K = \mathbb{Q}(\mu_n)$. Show that μ_n generates $\mathcal{O}_K$ as an abelian group and that $\mathcal{O}_K/\mathbb{Z}$ is unramified at all prime ideals not containing n!*

Problem 3 (3 points). *Let R be a Dedekind domain with field of quotients K and $|\cdot|$ an ultrametric absolute value on K such that $|x| \leq 1$ for all $x \in R$. Show that $\mathfrak{p} = \{x \in R \mid |x| \leq 1\}$ is a prime ideal of R.*

It follows from the definitions that $\mathfrak{p}$ depends only on the equivalence class of $|\cdot|$. In the case of the trivial absolute value we have $\mathfrak{p} = \{0\}$. The non-zero $\mathfrak{p}$ define a discrete valuation $K^\times \xrightarrow{v_\mathfrak{p}} \mathbb{Z}$ on K, sending x to the exponent of $\mathfrak{p}$ in the prime ideal decomposition of Rx. As always we put $v(0) = \infty$. It is well known from section 2 of the lecture that

$$(1) \qquad \begin{aligned} v(x+y) &\geq \min\big(v(x), v(y)\big) \\ v(xy) &= v(x) + v(y). \end{aligned}$$

Problem 4 (3 points). *If $K^\times \xrightarrow{v} \mathbb{Z}$ satisfies (1) and $B > 1$ is a real number, show that we have an ultrametric absolute value on K defined by*

$$|x| = B^{-v(x)}.$$

Of course the equivalence class of $|\cdot|$ does not depend on the choice of B. In addition, if $v = v_\mathfrak{p}$ as above then $|r| \leq 1$ for all $r \in R$. The previous two problems therefore define candidates for both directions of the bijection claimed in Proposition 3.1.2 in the lecture, and it remains to show that they are indeed inverse to each other. This is obvious in the case of the trivial absolute value and the zero ideal. It is also clear that in the the case of $\mathfrak{q} \in \mathrm{Spec} R \setminus \{\{0\}\}$, if we define $|\cdot|$ by putting $v = v_\mathfrak{q}$ in Problem 4 then the prime ideal $\mathfrak{p}$ from 3 equals $\mathfrak{q}$. The remainder of the proof of Proposition 3.1.2 of the lecture now follows from

Problem 5 (5 points). *Let $\|\cdot\|$ be a non-trival absolute value on K such that $\|r\| \leq 1$ for all $r \in R$. Let $\mathfrak{p}$ be given as in Problem 3. Show that $B = |x|$ does not depend on the choice of $x \in \mathfrak{p}^{-1} \setminus R$! Also, show that we have $|x| = \|x\|$ for the absolute value defined by Problem 4!*

Let K be equipped with an ultrametric absolute value $|\cdot|$. By a *ball* in K we understand a subset of the form

$$B = \{\xi \in K \mid |x - \xi| \leq r\} \tag{2}$$

with $r \in [0, \infty)$ or

$$B = \{\xi \in K \mid |x - \xi| < r\} \tag{3}$$

with $r \in (0, \infty)$. If similar definitions are applied to the ordinary absulute value on the field of real or complex numbers, then the center x is uniquely determined by the subset $B \subseteq K$. This is not the case here. Instead, we have the following interesting observation:

Problem 6 (4 points)**.** *For* (2) *and* (3)*, show that B does not change if x is replaced by another element of B! Show that for two balls $B_{1,2}$ in a field K equipped with an ultrametric absolute value, one the four relations $B_1 = B_2$, $B_1 \subset B_2$, $B_2 \subset B_1$ or $B_1 \cap B_2 = \emptyset$ always holds.*

Five of the 25 points available from this exercise sheet are bonus points which are disregarded in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted to the tutor by e-mail before Friday December 13 24:00.