Problem sheet 8 Rigid analytic geometry Winter term 2024/25

Problem 1 (3 points). *Let $(X_i)_{i\in I}$ be Priestley spaces. Put $X = \prod_{i\in I} X_i$ equipped with the product topology and the partial order where $(x_i)_{i\in I} \preceq (\xi_i)_{i\in I}$ if and only if $x_i \preceq \xi_i$ for all $i \in I$. Let $X \xrightarrow{p_i} X_i$ be the projection to the i-th factor, which obviously is a morphism of Priestley spaces. Show that for every Priestley space T, we have a bijection*

$$\operatorname{Hom}(T, X) \xrightarrow{\cong} \prod_{i\in I} \operatorname{Hom}(T, X_i)$$

$$f \to (p_i f)_{i\in I}$$

where Hom *is taken in the category of Priestley spaces.*

Thus, X is the product of the X_i in the category of Priestley spaces.

Problem 2 (2 points). *In the situation of the previous problem, show that the spectral topology X^s on the Priestley space X is the product topology $\prod_{i\in I} X_i^s$.*

It follows that products $\prod_{i\in I} Y_i$ of spectral spaces exist and coincide with the ordinary products of topological spaces.

Problem 3 (2 points). *Let $X \xrightarrow{f,g} Y$ be morphisms of Priestley spaces and*

$$K = \{x \in X \mid f(x) = g(x)\}$$

equipped with the induced topology and the restriction of the partial order from X. When T is a Priestley space, show that

$$\operatorname{Hom}(T, K) = \{t \in \operatorname{Hom}(T, X) \mid ft = gt\}.$$

Thus, K is the equalizer of f and g in the category of Priestley spaces.

Problem 4 (2 points). *In the situation of the previous problem, show that K^s, the spectral space obtained from the Priestley space structure on K, is K with the topology induced from X^s.*

It follows that equalizers in the category of spectral spaces exist and coincide with the subset where the two spectral maps coincide, equipped with the induced topology.

For the following problems, one bonus point is awarded when the example given is nat and another one when it is a Tate ring in the sense of Huber. Your claims that your solution satisfies this condition must be correct to get the point but it is not necessary to prove this claim.

Problem 5 (2+2 points)**.** *Give an example of an unbounded topologically nilpotent subset of a topological ring.*

In particular, such a subset will fail to be power bounded.

Problem 6 (2+2+1 points)**.** *Give an example of topologically nilpotent subsets X and Y of a topological ring such that $X \cup Y$ fails to be topologically nilpotent. The third bonus point is for examples where one of the two subsets is also bounded.*

For the following problems let A be a Tate ring and $(A^\sharp, s)$ a pair of definition.

Problem 7 (1 point)**.** *Show that $A = \bigcup_{n\in\mathbb{Z}} s^n A^\sharp$!*

Problem 8 (2 points)**.** *Let $U \subseteq A$ be a neighbourhood of zero. Show that there is a natural number n such that $s^n A^\sharp \subseteq U$!*

In other words $\left\{s^n A^\sharp \;\middle|\; n \in \mathbb{N}\right\}$ is a neighbourhood base of 0 in A.

Problem 9 (5 points)**.** *For a subset $X \subseteq A$, show that the following conditions are equivalent:*

- *X is bounded.*
- *$X \subseteq s^n A^\sharp$ for some $n \in \mathbb{Z}$.*
- *There is a finitely generated $A^\sharp$-submodule $M \subseteq A$ containing X.*

Problem 10 (3 points)**.** *For a subset $U \subseteq A$, show the equivalence of the following conditions:*

- *U is a neighbourhood of 0.*
- *There is a natural number n such that $s^n A^\sharp \subseteq U$.*
- *For every bounded subset X there is a natural number n such that $s^n X \subseteq U$.*
- *For every finitely generated $A^\sharp$-submodule $M \subseteq A$ there is a natural number n such that $s^n M \subseteq U$.*

Nine of the 29 points from this sheet are bonus points which are not counted in the calulation of the 50%-threshold for passing the exams.

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday December 16.