AG Bonn-Wuppertal-Düsseldorf-Bochum, WS 2005/06: Exotic spheres and the Kervaire invariant 1 problem

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Introduction

There are essentially three parts in this program. The first part is geometrical in nature, classical and one finds many good references in the literature. It is about surgery theory and the work of Kervaire-Milnor on exotic spheres. A good survey about this is the Harvard senior thesis of Joshua Plotkin available on the web page *http://www.maths.ed.ac.uk/∼aar/surgery/*.

This leads us to the second part, which still contains some geometry, but is much more of homotopy theoretic nature. In this part, we examine the reduction (due to Browder) of the indeterminacy left in the picture of exotic spheres to a purely homotopy theoretic problem. Except for the talk 8, which is based on Browder's (difficult) paper, the references we have chosen for this part are quite readable.

Finally, we try to give an overview of the most striking results on the Kervaire-invariant-one problem. In this part, most of the talks are very challenging (especially talk 11 and talk 12), the reason being that these results are either more recent (and have not been reworked and simplified) or require intricate computations.

The plan in more details

In the first part, we examine the problem of the existence and classification of exotic smooth structures on the topological spheres S^n. A homotopy sphere (*h-sphere*) is a smooth manifold homotopy equivalent to the ordinary sphere. The set Θ^n of h-cobordism classes of oriented h-spheres of dimension n is an Abelian group under connected sum. The h-cobordism theorem allows us to equate Θ^n with *the set of smooth structures on* S^n provided $n \geq 5$. Using the Thom-Pontryagin construction, one establishes an exact sequence

$$0 \longrightarrow bP^{n+1} \longrightarrow \Theta^n \xrightarrow{K} \pi_n^S / \operatorname{im} J_n = \operatorname{coker} J_n$$

where bP^{n+1} is the group of oriented h-spheres of dimension n bounding a parallelizable manifold, π_n^S is the n^{th} stable homotopy group of spheres and $J_n : \pi_n O \longrightarrow \pi_n^S$ is the J-homomorphism from the homotopy groups of the infinite orthogonal group to π_n^S. The map K is defined because any homotopy sphere is stably parallelizable.

This sequence is analyzed with the help of surgery theory and one shows that the map $K : \Theta^n \longrightarrow \operatorname{coker} J_n$ is at most $\mathbb{Z}/2$ in dimensions $n = 4k+2$. This $\mathbb{Z}/2$ indeterminacy corresponds to a surgery obstruction -*the Kervaire invariant*- and the realizability of this obstruction by a manifold is called the *Kervaire invariant 1 problem.*

In the second part, we study the fundamental results of Browder on the Kervaire invariant 1 problem. He proved that :

1. the Kervaire obstruction vanishes in dimensions $n = 4k + 2 \neq 2^{i+1} - 2$,
2. there exists a manifold of Kervaire invariant 1 in dimension $2^{i+1} - 2$ if and only if there exists a stable map θ_i detected by the class h_i^2 in the classical mod 2 Adams spectral sequence (in the Adams spectral sequence language : there are maps of Adams filtration 2 detected by h_i^2).

It is known that there exists maps θ_i for $1 \leq i \leq 5$. The maps θ_1, θ_2 and θ_3 are easy to construct. The existence of θ_4 is more involved, but there are many (independent) constructions available in the literature, using geometry, homotopy, or both. The construction of θ_5 is difficult and though there are at least three independent accounts for such a construction, all are purely homotopy theoretic in nature, and very subtle. The problem is still completely open in higher dimension, and is usually considered as difficult.

The aim of the third part of the AG is to make an overview of the most recent and striking result in the subject. We begin with the very elegant geometric construction due to J. Jones of θ_4. We will then explain how to construct Mahowald's η_j family, which consists of the only other elements having Adams filtration 2. This helps in to understand the Adams spectral sequence picture. We next consider the purely homotopy theoretic constructions of θ_4 and θ_5. Finally, we consider the result of Minami (one of the most recent new results in the subject) which gives a hint of how 'complicated' the elements θ_i are (if they exist!) : he shows that Kervaire maps $\theta_i, i \geq 5$ do not factor thought the double transfer if they happen to exist.

In a last talk, we give a global picture of what is known on the Kervaire invariant 1 problem, and the different approaches to solve it.

The talks in detail

In the following, some talks are labelled challenging talks. *By this, we mean, that these talks would require a great deal of work for someone who has not seen these things before.*

0. Intro (J. Ebert) We give a short introduction to the first part of the program.

1. Homotopy spheres (J. Wang) The basic constructions: Handle decompositions, Morse functions, surgery, cobordisms and connection between these things. The connected sum as a special case (for example [13], [18]

The h-cobordism theorem (without proof). Homotopy spheres. Proof of the Poincare conjecture in dimension $n \geq 6$ ([18], p.109). Definition and group structure of Θ^n ([12], p.505-508).

Plumbing: The construction of the Milnor and Kervaire manifolds and proof that their boundaries are homotopy spheres in dimensions $4k-1$ and $4k+1$, respectively. The main reference is [13], p. 118-23, see also [5], p.429-29, [2].

Remark: The homotopy spheres bounding the Milnor manifolds give example of exotic spheres (see next talk).

2. Homotopy spheres and stable homotopy groups of spheres (M. Castillo) A short introduction to : framed manifolds and framed cobordisms, Pontryagin's theorem that $\Omega_n^{fr} \cong \pi_n^S$, definition of the J-homomorphism.

Homotopy spheres are framed manifolds, the exact sequence $0 \to bP^{n+1} \to \theta^n \xrightarrow{K} \operatorname{coker} J_n$.

The signatures of almost parallelizable manifolds. The important statements are theorem 8.5 on p. 190 in [13], th. 8.7 on p.191. Also consult p.508-12 in [12] and [11]. Prove that the boundaries of the Milnor manifolds are exotic.

References : All this material is treated in chapter IX of [13]. For the Pontryagin construction, see also [19] or [5], p118-26.

3. Surgery I Motivation for the use of surgery in the study of Θ^n. The effect of surgery on homotopy and homology groups. Representation of elements in $\pi_r(M)$ by embedded spheres with trivial normal bundle. Framing a surgery. Surgery below the middle dimension.

Then explain the main result of surgery in odd dimensions (without proof), [13], 210-15 or [12],thm 6.6. The Whitney trick (sketch the proof) and the hard Whitney embedding theorem (without proof).

References : the main references are [13], p.195-202, see also [12], p. 512-14, and [14], p. 64-71. For the Whitney trick, see Milnor [18] and [23].

4. Surgery II (J. Ebert) First, one has to prove lemma 7.1 in [12], p.526 alias theorem 2.3 in [14].

On even-dimensional manifolds, it is not always possible to perform surgery. Lemma 7.1 in [12] gives a sufficient criterion. Explain how to satisfy this criterion.

For dimensions $n = 4k$, one is led to the study of the signature and the classification of quadratic forms over $\mathbb{Z}$. Prove that the obstruction is a bordism-invariant. Computation of bP^{4k} and proof of $\operatorname{im} P_{4k} = \operatorname{coker} J_{4k}$. [13], 202-06, [14] 71-79, [2], p.52-56 [12], 526-31.

For dimensions $4k + 2$, one is led to quadratic forms over $\mathbb{Z}/2$ and their Arf invariant. Define the naive Kervaire invariant as the Arf invariant of a certain quadratic form. The Kervaire invariant 1 problem. Computation of bP^{4k+2} and $\operatorname{im} P_{4k+2}$ depending on the Kervaire-invariant problem. [13], 206-210, 215-19.

5. Browder's work I: Introduction (C. Roitzheim) The speaker should briefly explain Browder's results and give the basics on the structure of the Adams spectral sequence (ASS) and its structure. The points that should be covered are

- construction of the ASS (for the spheres), Adams filtration, convergence,
- identification of the E^2-term, and enough calculation to identify what $E_2^{1,*}$ and $E_2^{2,*}$ are,
- the fact that the spectral sequence carries products,
- the computation of the Adams differential $d_2 h_{i+1} = h_0 h_i^2$ for $i \geq 3$ (as in the book of Kochman, using power operations, see references below)

One should be careful not to go over time with this talk, because one could spend a whole semester on the ASS! The main point for us is to give sense to the the statement of Browder's theorem, and to have enough background on the ASS for what follows.

References: [3], the book of Kochman (*Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs, 7, AMS) is very useful. McCleary's user's guide can be helpful, in particular to get more references.

6. Browder's work II: Homotopy theoretic definition of the Kervaire invariant (J. Kuhr) All the material in [4] should be covered. Special attention should be given to

- the relation between Wu-spectra and quadratic forms,
- the *corollary 1.13* (identification of Brown's construction to the surgery obstruction),
- the *theorem 1.18* which states a *change of framing formula* that will be needed in talk 9.

Remark 1 : one can avoid to discuss Poincare duality spaces : we only need to consider smooth manifolds.

Remark 2 : The examples given by 1.25, 1.26, 1.27 are very enlightening, and might illustrate the discussion.

References: [4] is the main reference. [9] helps to understand what we need from [4], and also give a synthetic account for [4].

7. Browders's work III : Vanishing result (G. Gaudens) One has to explain

- the Kahn-Priddy theorem (without proof),
- the work of N. Ray (see references below),
- how these two yield the vanishing of the Kervaire obstruction in dimensions $n = 4k + 2 \neq 2^{i+1} - 2$.

Remark: The transfer map will be used twice more in the sequel and one should therefore spend some time in explaining what it is.

References: [24], [9], we leave it to the speaker to find an appropriate reference on the (classical) Kahn-Priddy theorem.

8. Browder's work IV : Reduction to homotopy theory (Ch. Ausoni) *(Challenging talk)*

Using [3] as main reference, the speaker should explain the homotopy theoretic reduction of the Kervaire invariant 1 problem : there exists a manifold of Kervaire invariant 1 in dimension $2^{i+1} - 2$ if and only if there exists a stable map θ_i detected by the class h_i^2 in the classical mod 2 Adams spectral sequence.

Remark: The proof has to be simplified a lot. Browder uses complicated definitions and very general constructions which we do not really need. Using Brown's definition of the homotopy theoretic definition of the Kervaire invariant, and by focusing on our special case, the difficulty in Browder's paper is concentrated in its last 2 sections. Even so, this uses secondary operation techniques and a great deal of calculations.

References : [3]

9. Jones' Geometric construction in dimension 30 The aim is to construct by geometric means a manifold of dimension 30 with Kervaire invariant 1. This means that one has to cover theorem A, B, C in [8].

References: the main reference is [8]

10. Infinite families Before trying to understand the homotopy theoretic approach, it seems good to have a good picture of the first two lines of the ASS. The aim of this talk is to construct Mahowald's family η_j, which is detected by the classes $h_1 h_j$ on the E_2-term of the ASS.

References : [6], [7]

11. Homotopy theoretic construction of θ_4 and θ_5 *(Challenging talk)*
The aim of this talk is to give the idea of the construction of θ_4 and θ_5 by purely homotopy theoretic means. This involves a good knowledge in manipulating Toda Brackets and their interpretation as obstruction to the existence of finite complexes.

The speaker has to make a careful choice on which details he wants to explain.

References : [1], [15], see also an appendix (due to Adams) of lectures of Browder in Princeton in 1967 (available on request).

12. Minami's theorem on double transfer (B. Schuster) *(Challenging talk)*
The transfer map $t : \Sigma^\infty B\mathbb{Z}/2 \longrightarrow S^0$ is epimorphic in stable homotopy in strictly positive dimensions. One can iterate the transfer by considering $t \wedge t$ followed by the multiplication $S^0 \wedge S^0 \longrightarrow S^0$. One can iterate transfers, and the more one iterates the transfer, the smaller is image the stable homotopy of spheres.

Mahowald had observed that the $\theta_i, i \leq 3$ factor thought the double transfer and this had lead to the hope that Kervaire invariant 1 elements, if they exist, would factor thought the double transfer. This would have meant that the Kervaire invariant one elements (if they exist) are somehow decomposable, and more tractable. This was disproved by Minami, using BP-operations technology. Lin and Mahowald gave later another proof based again on the classical ASS and explicit (difficult) computations.

The speaker should explain Minami's paper with enough detail to make it clear how one can use BP-operation techniques in this setting.

References : [16], [21]

Closing lecture : State of the conjecture, other approaches (G. Gaudens) The speaker should give an overview of what's known on the Kervaire invariant one problem, what are the different approaches to solve it.

References : The survey [22] is very useful.

References

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