A SEMINAR ON THE NON-EXISTENCE OF ELEMENTS OF KERVAIRE INVARIANT ONE (AFTER HILL, HOPKINS, AND RAVENEL)

ORGANIZED BY
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Form. As usual, the seminar will be structured into seven afternoon sessions. During each session there will be two talks. Each talk therefore will take

$$\frac{1}{2}(180 \text{ minutes} - \text{duration of coffee break}).$$

If in doubt, plan for 60 minutes and assume the questions from the audience will fill the remaining time.

Function. The aim of this seminar is to follow the main arguments given in [HHR] to show that certain elements h_j^2 on the E_2 page of the mod 2 Adams spectral sequence do not survive to E_∞ if $j \geqslant 7$. By classical results and the work of Barratt, Jones, and Mahowald these elements elements survive if $j \leqslant 5$. This leaves open the question of the survival of such an element only in degree $2^{6+1} - 2 = 126$.

Disclaimer. It is unlikely that there will be a complete proof of the main result at the end of the seminar, for various different reasons: Not only do the authors defer some foundational material on multiplicative equivariant stable homotopy theory to a still unwritten appendix. But even without the missing appendix, there is already now too much material to be covered. This prompted our decision not to proceed in the given order. In particular, we have placed the discussion of the reduction and slice theorems towards the end, as they are easy to state, but their proofs are rather involved.

Session 1: Applications and Overview 14/10/10

Talk 1.1: Applications

The main result of [HHR] answers a central question in algebraic topology and all the following talks will focus on its proof. However the solution of this problem has several applications to geometric topology and it would be a shame not to mention these.

There are many ways of telling this story, and the speaker is encouraged to find her or his own here. For example, one way to start is with the Pontryagin-Thom construction and Pontryagin's mistake in the computation of the second stable stem. This is how Hopkins proceeded in his talk at Atiyah's birthday conference [Hop].

Another approach would be to start with the classification of exotic spheres, in particular the subgroup bP^{n+1} of the group of h-cobordism classes of exotic n-spheres which consists of those classes which are representable as boundaries of parallelizable $(n+1)$-manifolds. Surgery theory reduces this to homotopy theory, and the Kervaire invariant question comes up in the case $n \equiv 2 \bmod 4$. At this point, the speaker could offer a glimpse into the deep and difficult work of Browder [Bro69] and Brown [Bro72]. However, a detailed exposition of these matters could easily occupy the majority of an entire seminar [EG05].

References: Apart from the sources already mentioned, there is Kervaire's original work [Ker60], and also a more recent exposition by Brown [Bro00]. You can read about exotic spheres in any textbook on manifolds and surgery theory, such as Lück's lecture notes [Lüc02].

Comments: This talk is appropriate for someone already familiar with the topic.

Talk 1.2: An overview of the proof

Welcome to homotopy theory! This talk will outline the proof of the main result: the existence of non-trivial differentials in the Adams spectral sequence. This can immediately be translated into a question about whether certain elements in the Adams-Novikov spectral sequence survive.

Over the course of this seminar we will show that these elements survive if and only if they survive in the Adams-Novikov spectral sequence for the C_8 homotopy fixed points of a particular C_8-equivariant spectrum. This is called the detection theorem and is proved by direct calculation.

The elements of interest must lie in the $2^i - 2$ stems of this spectrum. By showing that the homotopy groups of this spectrum are 256 periodic (the periodicity theorem) and that the -2-stem is trivial (the gap theorem and fixed point theorems) we will see that these elements do not exist for $i \geq 8$.

The proof of these results will require a number of tools from equivariant stable homotopy theory, most notably the slice filtration and the slice spectral sequence. The technical heart of [HHR] is the slice theorem which is used to prove the results above.

References: Section 1 in [HHR] is a great place to start, however a more in depth outline of the proof should be given. In particular a discussion of the role slice theorem and slice spectral sequence.

Session 2: Equivariant homotopy theory 28/10/10

It is best to assume that G is a finite group throughout the seminar.

Talk 2.1: Equivariant stable homotopy theory

The treatment of G-spectra in [HHR] is rather informal; but the authors say that orthogonal spectra will be used for foundations. This talk should give the necessary background from equivariant stable homotopy theory, with orthogonal spectra as the point-set model. In particular, it should be explained that the theory depends not only on the group G, but also on the choice of a G-universe – the extremes are naive and genuine universes. Topics that should be covered include

- G CW-complexes.
- Mackey functors.
- $RO(G)$-graded homotopy groups.
- Categorical formalities (change of universe, fixed points, etc.).

To help the following speakers, you may want to cover some examples such as Bredon spectra (based on Mackey functors), see 2.5, $M\mathbb{R}$-theory, see 4, and $K\mathbb{R}$-theory, which is optional, but fun.

Exercise: Can you tell the difference between these 'Real' theories and the standard equivariant complex Thom spectra/complex K-theory spectra?

References: Be sure that you cover 2.1 and 2.2 in [HHR], and possibly some parts of 2.3, 2.5 and 4. A good overview can be gained from the notes [Alaska], especially Chapters I, IX, X, XII, and XIII. See also [GM95], which is much shorter (on detail). The reference for orthogonal G-spectra is [MM02].

Comments: While it would help to have some knowledge of LMS-spectra or Orthogonal spectra, it is not a prerequisite for giving this talk. The same goes for unstable equivariant homotopy theory. This talk is great for someone who wants to become acquainted with the foundations of stable equivariant homotopy theory.

Talk 2.2: Bredon (co)homology and the cell lemma

Bredon cohomology is a fundamental topic in equivariant homotopy theory. The speaker should discuss the bijection between Mackey functors and coefficient systems for Bredon cohomology. It will be helpful to discuss G-cellular chains and cochains. As an application, the speaker should do the calculation for the cell lemma carefully. Although geometric fixed points will be discussed in detail in the next talk, it is recommended to say enough about them to prove Proposition 2.42.

A fun optional topic is to construct Bredon cohomology as a representable functor.

References: Section 2.6 of [HHR] and V.4 and XIII of [Alaska].

Comments: This talk will be one of the most elementary and can be given by a novice.

Session 3: Fixed points and products 11/11/10

Talk 3.1: Fixed points

This talk should explain the various concepts of fixed point spectra used in equivariant stable homotopy theory: categorical fixed points, homotopy fixed points, and geometric fixed points. The speaker should talk about the classifying spaces $E\mathcal{F}$ for a family $\mathcal{F}$ of subgroups and the square relating the various notions of fixed points with Tate cohomology.

Exercise: Give an example of a G-space X with $\Sigma^\infty(X^G) \not\simeq (\Sigma^\infty X)^G$?

After this the proof of the fixed point theorem should be given, at least up to a lemma stating $\Phi^H \bar{\mathfrak{d}}_k^H = 0$ which will be proven next week. The more you can say about these generators the easier you will make it for future speakers.

References: This is 2.4 and 11 in [HHR]. See also [Alaska] (XVI.1-XVI.3, XXI.1), and V.3 and V.4 in [MM02] for fixed points in the context of orthogonal spectra. See [HK01] for the $K\mathbb{R}$-theory version of the theorem.

Comments: This talk is appropriate for someone who wants a deeper understanding of equivariant stable homotopy theory. This talk is also recommended for those interested in algebraic K-theory computations, since the fixed point square discussed above frequently makes an appearance.

Talk 3.2: Multiplicative properties.

This talk will give an overview of the theory of orthogonal G-spectra following [MM02], as well as a discussion of the norm. The segment on orthogonal G-spectra should be focused on the construction of the symmetric monoidal smash product. The segment on the norm should follow section 2.3 in [HHR]. The central Propositions 2.12 and 2.13 about the norm appear without proof, so these need only be stated and motivated.

References: Cover 2.3 in [HHR]. Maybe [GM97] helps to get some idea about the norm.

Session 4: $M\mathbb{R}$-theory and the slice spectral sequence 25/11/10

Talk 4.1: $M\mathbb{R}$-theory, equivariant formal group laws, and generators

This talk will briefly cover Real cobordism theory, equivariant formal group laws, and the choices of generators for $\pi_* MU^{((G))}$. The goals are to

(1) understand the universal property $\pi_* MU^{((G))}$ has with respect to equivariant formal group laws for the proof of 12.4.i (although the proof of 12.4 will be given later),
(2) prove that $\Phi^H \bar{\mathfrak{d}}_k^H = 0$ to complete the proof of the fixed point theorem,
(3) to state the required results about generators for the computations in the slice spectral sequence, especially Proposition 4.51.

References: Use [HK01] and Sections 4.1, 4.3, and 12 in [HHR].

Comments: This is a good talk for someone interested in complex oriented cohomology theories and their equivariant analogues. It is helpful to have some experience with the classical material on complex oriented cohomology theories (see [Rav00] or [Ada74]).

Talk 4.2: The slice spectral sequence

This talk should introduce slices, the slice tower, and the resulting slice spectral sequence. This can be thought of as an equivariant analogue of the Postnikov decomposition, but with the G-cells replaced by slice cells. A great deal of this work is formal and appears in the construction of many spectral sequences. The theory gains its power from Theorem 3.38, which implies convergence and vanishing regions.

Rather than losing too much time on the formal aspects, the emphasis should be on arguments relevant to the proof of the convergence and vanishing results.

References: Section 3 in [HHR]. A brief look into [Dug05] might be inspiring as well.

Comments: This is a good talk for someone interested in learning about the slice spectral sequence but has some understanding of the formal construction of spectral sequences, for example, the construction of a spectral sequence from an exact couple.

Session 5: Applications of the slice theorem and computations
9/12/10

Talk 5.1: Consequences of the slice theorem

In this talk the speaker will state the slice theorem, postponing the proof for later, and cover some immediate consequences. First the speaker should use this and the cell lemma to prove the gap theorem. In the second segment the speaker should identify the E_2-term, through a range, of the slice spectral sequence for $\pi_* MU^{((G))}$. Be sure to recall the necessary facts and and notation from Section 4.3.

References: The proof of the gap theorem is spelled out in Section 9 of [HHR]. The identification of the E_2-term of the slice spectral sequence is given in 10.1 of [HHR].

Comments: This is a great talk for someone interested in cohomology computations. Since the talks on this day are tightly bound together, it is probably a good idea to work with the other speaker in preparing this talk.

Talk 5.2: Differentials in the slice spectral sequence and the periodicity theorem

This talk should prove the slice differential theorem, the periodicity theorem, and complete the proof of the fixed point theorem. Using the identification of the E_2-term from the previous talk and the results about generators from the talk on $M\mathbb{R}$-theory, and the unproven Proposition 7.3, the speaker should prove the differential theorem. From this point it is straightforward to prove the periodicity theorem.

There are a number of pitfalls to avoid in this talk. Try to be careful about your indexing, notation, and when you are working with G or H fixed points.

References: Section 10 in [HHR].

Comments: This is a great talk for someone interested in cohomology computations and has worked with non-degenerate spectral sequences before. Since the talks on this day are tightly bound together, it is probably a good idea to work with the other speaker in preparing this talk.

Session 6: The detection and slice theorems 13/1/11

Talk 6.1: The detection theorem

In this talk the speaker will construct a graded formal group law which canonically receives an equivariant classifying map from $\pi_* MU^{((C_8))}$. Accepting the computations at the end of section 12 as given, the speaker should show that there is a map respecting a valuation from the 2nd line of the Adams-Novikov E_2 term to the cohomology of C_8 with coefficients in R_*. He or she should say just enough about the chromatic spectral sequence to define a valuation on the source and evaluate it on the elements in question. Proving 12.4 should be straightforward from the previous talk on equivariant formal group laws and the computations at the end of section 12. The computation of the cohomology of the target is easy and can be omitted if there is not enough time.

The speaker should be aware that there is no analogue of the chromatic spectral sequence for the cohomology of the target because the formal group law on R_* is not Landweber exact.

References: Section 12 in [HHR]. The chromatic spectral sequence is covered in Chapter 5 of [Rav00].

Comments: This talk is good for someone interested in computations of homotopy groups. Since so many results are classical and the computations are elementary this is a good talk for a novice.

Talk 6.2: The slice theorem

The goal of this talk is to expand the outline of the proof of the slice theorem in Section 8 of [HHR]. One should explain the inductive method of argument by defining the intermediate spectra, listing their properties (which we have not proven in this seminar yet), and outlining the proof of 8.1 provided there. Be sure to draw attention to the role of the reduction theorem in the proof of 8.1. The attentive members of the audience should walk away with an understanding of how the slice theorem is to be proved and, most importantly, what lemmas we still need to complete the proof. If you can prove some of these lemmas, especially those from Section 5, next weeks speakers will appreciate it.

References: Primarily Section 8 and the introduction to Section 6 in [HHR], but these sections depend heavily on material from Sections 4 and 5.

Comments: Giving this talk will require covering material that is not in Section 8 nor the introduction to Section 6. The paper assumes a number of results and definitions that we have not covered yet in this seminar. Although you should give the definitions you will probably end up pushing many proofs on to the speakers for next week who will, most likely, be unable to complete all of them.

Session 7: The reduction theorem 27/1/11

Talk 7.1: The reduction theorem

The goal of this talk is to cover the material in Section 6.1 and the proof of Proposition 7.3 (which we needed for the periodicity theorem). Since Section 5 involves properties of the slice filtration special to the case $G = C_{2^n}$ it might be instructive to cover those proofs in depth.

References: Sections 5, 6, and 7 in [HHR].

Talk 7.2: The reduction theorem continued

The goal of this talk is to continue the previous speakers exposition of Section 7 and complete the proof of the reduction theorem. Since this is most likely an impossible task for a single talk, try to leave the details for the end and the interested attendees.

References: Section 7 of [HHR].

References

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