HOMOTOPY & COHOMOLOGY
GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

## Winter School

# "From Field Theories to Elliptic Objects"

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## Talk No. 8

## Speaker: Ferit Deniz

Super manifolds Ferit Deniz

**Def** A *ringed space* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of commutative rings on it.

**Ex** Let $M^n$ be a smooth mfd. Let $\mathcal{C}^\infty_M$ be the sheaf of smooth functions on $M$. Then $(M, \mathcal{C}^\infty_M)$ is a ringed space.

**Def** Let $X, Y$ be ringed spaces. A *morphism* $f: X \longrightarrow Y$ is a pair $(|f|, f^*)$ where $|f|: |X| \longrightarrow |Y|$ is a continuous map

and $f^*: \mathcal{O}_Y \longrightarrow |f|_* \mathcal{O}_X$

is a homomorphism of sheaves on $Y$

( this is a collection of algebra homomorphisms

$$f^*_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(|f|^{-1}(U))$$

compatible with restrictions.)

The ringed space $(M, \mathcal{C}^\infty_M)$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$. Conversely each ringed space $M$ which is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is a smooth mfd (+ additional properties: $|M|$ is hausdorff and second countable)

## Super ringed spaces

$X = (|X|, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of super commutative algebras

Example ($\mathbb{R}^{p|q}$) The underlying space is $\mathbb{R}^p$ and the sheaf $\mathcal{O}_{\mathbb{R}^{p|q}}$ is

$$\mathbb{R}^{p|q}(U) = \mathcal{O}_{\mathbb{R}^{p|q}}(U) = \mathcal{C}^\infty(U)[\vartheta^1, \dots, \vartheta^q]$$

(= the exterior algebra over $\mathcal{C}^\infty(U)$).

(these are super algebras by giving $\mathcal{C}^\infty(U)$ the degree $0$ and $|\vartheta^j| = 1$)

For a super ringed space $X$ we denote the subsheaf of even sections by $\mathcal{O}_X^{ev}$

— " —— odd —— $\mathcal{O}_X^{odd}$

Def (super manifold)

Let $|X|$ be a second countable hausdorff space. Then a super ringed space $X = (|X|, \mathcal{O}_X)$ is called a super manifold (of dim. $(p|q)$) if it is locally isomorphic to $\mathbb{R}^{p|q}$.

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notation $u \in \mathcal{C}^\infty(U)[\vartheta_1, \dots, \vartheta_q]$ :

$$u = u^0 + \sum_{|I|>0} f_I \vartheta_I$$

$u^0, f_I \in \mathcal{C}^\infty(U)$, $\vartheta_I = \vartheta_{i_1} \cdots \vartheta_{i_k}$ $\quad i_1 < \dots < i_k$)

**Example** Let $X^p$ be a smooth mfd and $E \to X$ a vector bundle of rank $q$. Let $\mathcal{O}_X$ be the sheaf of (locally) smooth sections of the exterior-algebra-bundle

$$\Lambda(E) \longrightarrow M.$$

This is locally isomorphic to $\mathbb{R}^{p|q}$
(if $X = \mathbb{R}^p$ and $E = X \times \mathbb{R}^q$, then

$$\mathcal{O}_X(U) = \mathcal{C}^\infty(U) \otimes \Lambda_{\mathbb{R}}(\mathbb{R}^q) \quad )$$

**Remark** This construction defines a functor

$$(\text{real vector bundles}) \longrightarrow (\text{super manifolds})$$

- one can prove that each smfd is isomorphic to such an smfd
- the isomorphism is not cononical.
- the morphisms in the image of the above functor preserve not only the $\mathbb{Z}/2$ grading but also the finer $\mathbb{Z}$ grading coming from the exterior bundle. But there are morphisms not preserving the latter. This means that the above functor is not full.

(we will see an example later)

# The underlying classical manifold

$X$ a smfd.

- we want to define a smooth structure on the topological mfd $|X|$.
- The smooth mfd with this structure will sometimes be called $X_{red}$.
- The dimension of $X_{red}$ will be $p$ where $\dim X = (p|q)$
- The construction $X \longrightarrow X_{red}$ should be functorial.

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Let $\mathcal{J}_X \subset \mathcal{O}_X$ be the ideal generated by nilpotent sections. Let $\tilde{\mathcal{O}}_X$ be the quotient sheaf $\mathcal{O}_X / \mathcal{J}_X$ (a ringed space)

Example For $X = \mathbb{R}^{p|q}$, $\mathcal{J}_X$ is the subsheaf generated by $\vartheta_1, \dots, \vartheta_q$ and hence

$$\mathcal{O}_X / \mathcal{J}_X \cong \mathcal{C}^\infty_{\mathbb{R}^p}.$$

This means that the ringed space $(\mathbb{R}^p, \tilde{\mathcal{O}}_{\mathbb{R}^p})$ is just $\mathbb{R}^p$ with the usual smooth structure.

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This has the consequence that

$$X_{red} := (|X|, \tilde{\mathcal{O}}_X)$$

is a smooth manifold

## $\tilde{\mathcal{O}}_X$ as a sheaf of functions :

let $u \in \mathcal{O}_X(U)$. For each $x \in U$ we can define a "value" $\tilde{u}(x) \in \mathbb{R}$ in the following way. $\tilde{u}(x)$ is the unique number such that

$$u - \tilde{u}(x) \in \mathcal{O}_X$$

is not invertible in each neighborhood of $x$ contained in $U$.

To see that $\tilde{u}(x)$ is indeed unique with this property, choose $V \subset U$ such that

$$\mathcal{O}_X|_V \cong \mathbb{R}^{p|q}$$

and write $\quad u = u^0 + \sum_{|I|>0} f_I \,\theta_I$.

Then check that $\tilde{u}(x) := u^0(x)$ has this property.

The functions of the form $\tilde{u}$ define a sheaf of commutative rings on $|X|$.

This sheaf is isomorphic to $\tilde{\mathcal{O}}_X$.

# Constructions by gluing

Let $|X| = \bigcup_i X_i$ where $\{X_i\}_i$ is a collection of open subsets of $X$, **closed under intersections** ($\forall i,j\ \exists k : X_i \cap X_j = X_k$)

Let for each $i,j$ with $X_i \subset X_j$ be given isomorphisms $f_{ij} : \mathcal{O}_j|_{X_i} \xrightarrow{\cong} \mathcal{O}_i$.

A sheaf $\mathcal{O}$ on $X$ together with isomorphisms

$$f_i : \mathcal{O}|_{X_i} \xrightarrow{\cong} \mathcal{O}_i$$

is called a *gluing* of the $\mathcal{O}_i$'s if

$$X_i \subset X_j : \quad \begin{tikzcd} \mathcal{O}|_{X_i} \arrow[rr, "f_j|_{X_i}"] \arrow[dr, "f_i"'] & & \mathcal{O}_j|_{X_i} \arrow[dl, "f_{ji}"] \\ & \mathcal{O}_i & \end{tikzcd}$$

**Theorem** There is a gluing of the $\mathcal{O}_i$'s if and only if:

- $f_{ii} = id$ on $\mathcal{O}_i$
- $f_{ki} = f_{kj} f_{ji}$ on $\mathcal{O}_k$

**Application** Existence of products in the Category of super mfd's.

(first: on rectangles and then glue together)

## Functor of points

by the Yoneda Lemma we have an embedding

$$\begin{aligned} S\mathcal{M} &\longrightarrow \mathrm{Set}^{S\mathcal{M}^{op}} \\ X &\longmapsto \mathrm{Hom}(-, X) \end{aligned}$$

such that the functor contains the same information as $X$. For each s-mfd $S$ we call $\mathrm{Hom}(S, X)$ the <u>$S$-points</u> of $X$ and denote it by $X(S)$.

<u>Examples</u> • $\mathbb{R}^{p|q}(S) \cong \left\{ (f_1, \dots, f_p, \varphi_1, \dots, \varphi_q) \;\middle|\; \begin{array}{l} f_i \in \Gamma(\mathcal{O}_S(S)^{ev}) \\ \varphi_j \in \Gamma(\mathcal{O}_S(S)^{odd}) \end{array} \right\}$

(see next theorem)

• Group structure on $\mathbb{R}^{1|1}$:

$$\begin{aligned} \mathbb{R}^{1|1}(S) \times \mathbb{R}^{1|1}(S) &\longrightarrow \mathbb{R}^{1|1}(S) \\ ((t, \vartheta), (t', \vartheta')) &\longmapsto (t + t' + \vartheta \cdot \vartheta', \vartheta + \vartheta') \end{aligned}$$

• $\mathcal{M}(\mathbb{R}^{0|0}) = |\mathcal{M}|$

• $GL(p|q)$

for each $S$ write the coordinates of $\cancel{\mathbb{R}^{p|q}}$

$$\mathbb{R}^{p^2+q^2 \mid 2pq}(S)$$

as a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

with $\left\{ \begin{array}{ll} A, D & \text{even, invertible} \\ B, C & \text{odd} \end{array} \right\}$ matrices.

# Definition of morphisms by local coordinates

**Thm** Let $X$ be a super mfd and
$f_1, \dots, f_p \in \mathcal{O}_X(X)^{even}$
$\varphi_1, \dots, \varphi_q \in \mathcal{O}_X(X)^{odd}$

Then for each $U \subset \mathbb{R}^{p|q}$ (with coordinates $t_1, \dots, t_p, \vartheta_1, \dots, \vartheta_q$) we have a unique morphism
$$\psi : M \longrightarrow \mathbb{R}^{p|q}$$
such that $\psi^*(t_i) = f_i \quad , \quad \psi^*(\vartheta_j) = \varphi_j$

proof of existence: By classical theory there is a unique smooth map
$$\tilde{\psi} : X_{red} \longrightarrow \mathbb{R}^p$$
such that $\tilde{\psi}^*(t_i) = \tilde{f}_i$

Now we have to define a homomorphism
$$\psi_V^* : \mathcal{C}^\infty(V)[\vartheta_1, \dots, \vartheta_q] \longrightarrow \mathcal{O}_X(\tilde{\psi}^{-1}(V))$$
for each $V \subset \mathbb{R}^p$. Assume that $X = \mathbb{R}^{m|n}$
$(\mathcal{O}_X(U) = \mathcal{C}^\infty(U)[x_1, \dots, x_n], \quad U \subset \mathbb{R}^n)$
let $F \in C^\infty(V)$. Then define $\psi_V^*(F)$ by
$$\psi_V^*(F) = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha F(f_1^\circ, \dots, f_p^\circ) \cdot (f - f^\circ)^\alpha$$
This is a finite sum since $f_i - f_i^\circ$ are nilpotent.

For uniqueness we need some preperation:

Let $X = (|X|, \mathcal{O})$ be a smfd and $p \in |X|$. Define

$$I_p := I_{p,X} := \{ [f] \in \mathcal{O}_p \mid f^0(p) = 0 \text{ in } \mathbb{R} \}$$

This is an ideal of $\mathcal{O}_p$.

Let $U \cong \mathbb{R}^{p|q}$ be an coordinate superdomain around $p$, and $x^i, \xi^j$ the pullbacks of $t^i, \vartheta^i \in \mathbb{R}^{p|q}(\mathbb{R}^p)$.

<u>Lemma</u>

(1) $I_p$ is generated by $[x^i]_p$, $[\xi^j]_p$.
For a morphism $\varphi: X \longrightarrow Y$ of smfds and any $k \geq 0$ we have

$$\varphi^*(I^k_{\tilde{\varphi}(x),N}) \subset I^k_{x,M}$$

(2) If $k > q$ and $f$ is a section around $p \in |X|$ such that $[f]_x \in I^k_x$ for all $x$ in some neighborhood of $p$, then $[f]_p = 0$

(3) For any $k \geq 0$ and any section $f$ around $p$ there is a polynomial $P$ in the $[x^i], [\xi^j]$ such that

$$f - P \in I^k_p$$

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Proof For a smooth function $g$ of $t_1, \dots, t_p$ use the taylor expansion

$$g(t) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha g(0) \cdot t^\alpha + \frac{1}{k!} \sum_{|\alpha| = k+1} t^\alpha \, g_\alpha(t)$$

$$g_\alpha(t) = \int_0^1 (1-s)^k \partial^\alpha g(s \cdot t) \, ds$$

proof of uniqueness

let $\psi$ and $\varphi$ two morphisms $X \longrightarrow \mathbb{R}^{p|q}$ such that $\psi^*(t^i) = \varphi^*(t^i)$ and $\psi^*(\vartheta^j) = \varphi^*(\vartheta^j)$

$\Rightarrow \quad \tilde{\psi} = \tilde{\varphi} : |X| \longrightarrow \mathbb{R}^p$

We must prove that $\psi^*(u) = \varphi^*(u)$ for all $u \in \mathcal{C}^\infty(U)[\vartheta^1, \dots, \vartheta^q]$, $U \subset \mathbb{R}^p$.

This is true for all polynomials in $t^i, \vartheta^j$.

Now let $u \in \mathcal{C}^\infty(U)[\vartheta^1, \dots, \vartheta^q]$ and write

$$g = \psi^*(u) - \varphi^*(u) \in \mathcal{O}_X(\tilde{\psi}^{-1}(U))$$

Let $k = m+1$ where $\dim X = m|n$ and let $x \in |X|$, $y := \tilde{\psi}(x) = \tilde{\varphi}(x) \in \mathbb{R}^p$.

by (3) of the Lemma there is a pol. $P$ in the $t^i, \vartheta^j$ such that $[u]_y = [P]_y + [R]_y$ where $[R]_y \in I_y^k$. Since $\varphi^*(P) = \psi^*(P)$ we have $g = [\psi^* R - \varphi^* R]_y \in I_x^k$ (by (1)

Since $x$ was arbitrary, we have $g = 0$ by (2)

# Some super geometry

The <u>tangent bundle</u> $\mathcal{T}X$ of a super mfd $X$ is defined to be the sheaf of (graded) derivations on $\mathcal{O}_X$.

<u>Example</u> $X = \mathbb{R}^{p|q}$

$\frac{\partial}{\partial x_i}$ is the derivation defined by

$$\frac{\partial}{\partial x_i}\Big( \sum_I f_I \vartheta_I \Big) = \sum_I \frac{\partial}{\partial x_i} f_I \cdot \vartheta_I$$

This is an even derivation.

$\frac{\partial}{\partial \vartheta_j}$ is the derivation defined by

$$\frac{\partial}{\partial \vartheta_j}\big( \mathcal{C}^\infty(U) \big) = 0 \quad , U \subset X \text{ open}$$

and $\frac{\partial}{\partial \vartheta_j}(\vartheta_i) = \delta_{ij}$

This is an odd derivation.

<u>Theorem</u> $\mathcal{T}\mathbb{R}^{p|q}$ is free of rank $p|q$ over $\mathcal{O}_{\mathbb{R}^{p|q}}$ with basis $\left\{ \frac{\partial}{\partial x_i} , \frac{\partial}{\partial \vartheta_j} \right\}$

<u>proof</u> (sketch)

Let $D$ be a derivation on $\mathbb{R}^{p|q}$ and rewrite the coordinates $x_1, \dots, x_p, \vartheta_1, \dots, \vartheta_q$ as

$y_1, \dots, y_m \quad , \quad m = p+q$.

Let $a_j := D y_j$

we want to show that

$$D = \sum a_j \frac{\partial}{\partial y_j} .$$

So define the derivation

$$D' = D - \sum a_j \frac{\partial}{\partial y_j}.$$

Now we have $D'(y_j) = 0 \quad \forall j$.

Hence $D'(P)$ for each polynomial in $y_i$.

You can apply the Lemma about polynomial approximation to deduce $D = 0$.

Linear independence is clear. $\square$

This has the consequence that $TX$ is a vector bundle over $X$:

<u>Def</u> Let $X$ be a smfd. A vector bundle of rank $(n,m)$ $V$ over $X$ is a sheaf of $\mathcal{O}_X$-modules which is <u>locally</u> free of rank $(n,m)$.

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The <u>cotangent bundle</u>:

$$\Omega^1(X) := T^*X := \mathcal{H}om_{\mathcal{O}_X}(TX, \mathcal{O}_X)$$

$n$-<u>Differential forms</u> are just sections of

$$\Omega^n(X) := \Lambda^n \Omega^1(X) .$$

For $X = \mathbb{R}^{p|q}$ define the 1-forms $dx_i$, $d\vartheta_j$ as the duals of $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \vartheta_j}$.

You can extend $d$ to all differential forms by linearity and Leibniz formula.

This gives us the de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1 X \to \Omega^2 X \to \dots$$

## super manifolds with boundary

local model: $\mathbb{R}^{p|q}_{\geq 0} := \mathbb{R}^{p|q}|_{\mathbb{R}^p_{\geq 0}}$

the <u>boundary</u> $\partial X$ of a s-mfd $X$ is a super mfd $\partial X = (\partial|X|, \mathcal{O}_{\partial X})$ with a morphism $\partial X \longrightarrow X$ over the inclusion $\partial|X| \hookrightarrow |X|$, with $\dim \partial X = (p-1, q)$ if $\dim X = (p|q)$.

<u>Example</u> $\partial \mathbb{R}^{p|q} = \mathbb{R}^{p-1|q}$
with the embedding

$$\mathbb{R}^{p-1|q}(S) \longrightarrow \mathbb{R}^{p|q}, \ (t_1, \dots, t_{p-1}, \vartheta_1, \dots, \vartheta_q) \longmapsto (t_1, \dots, t_{p-1}, 0, \vartheta_1, \dots, \vartheta_q)$$

<u>Definition</u> (Metric structures on $(1|1)$-smfds)
Let $X$ be a $(1|1)$-smfd. A 1-form $\omega$ on $X$ is called a metric on $X$ such that $\omega|_{|X|}$ and $d\omega|_{|X|}$ are nowhere vanishing and such that the Berezin integral of $\omega$ over $(0|1)$-dimensional sub mfds is positive.
Such an $\omega$ induces a metric structure on $|X|$ in the usual sense.

<u>Example</u> $\omega = dt + \vartheta\, d\vartheta$ on $\mathbb{R}^{1|1}$

for $(t, \vartheta) \in \mathbb{R}^{1|1}$ the translation

$$T_{t,\vartheta} : \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1}$$
$$(z, \eta) \longmapsto (t + z + \vartheta\eta, \ \vartheta + \eta)$$

preserves the metric structure $\omega$