Winter School

From Field Theories to Elliptic Objects
Schloss Mickeln, February 28 — March 4, 2006
Graduiertenkolleg 1150: Homotopy and Cohomology

Prof. Dr. G. Laures, Universität Bochum
Dr. E. Markert, Universität Bonn

Details : http://www.math.uni-bonn.de/people/GRK1150 → study program

Topic

In the past two decades mathematicians started to investigate the geometric properties of manifolds by looking at their ambient loop spaces. The stimulation came from particle physists who analyzed these infinite dimensional objects heuristically. The results are very mystifying since they connect physics and differential topology with the theory of elliptic curves and modular forms.

An explanation is expected to come from a new cohomology theory which should be regarded as a higher version of topological K-theory. Such a theory can be obtained by methods of algebraic topology but its relation to loop spaces and field theories is still unclear.

Graeme Segal and Dan Quillen gave a first description of the elements in this new theory in terms of field theories. Later their approach was modified and improved by Stefan Stolz and Peter Teichner. The seminar gives an overview of their work while mainly focusing on the relationship between 1-dimensional euclidean field theories and classical K-theory which is now understood.

Program

Tuesday, February 28, 2006

10:30-12:00 (1) **Elliptic Cohomology and Conformal Field Theories: An Introduction.**
(Prof. G. Laures)

14:00-15:00 (2) **Field Theories I.**

- Overview: Quantization of classical systems, path integrals, Feynman Kac formula, Wiener measure
- Example: Heat operator, Laplace operator.
- Overview: Quantization of fields (strings): Heuristic path integral formula, problems

Literature: Primary: [Segal], [Stolz-Teichner], [Rabin]; Secondary: [Gawedzki-1989], [Woodhouse], [Peskin-Schroeder], [Ramond], [Zee], [Gawedzki-1999], [Teichner]

15:30-16:30 (3) **Field Theories II.**

- σ-model: field theories in manifolds
- Functorial definition of field theories of dimension d. Geometrical information: E/C/T-FT. Spaces of functors. Field theories on a space X.
- The previous example (heat operator) in this language.

Literature: Primary: [Segal], citeST, [Rabin]; Secondary: [Gawedzki-1989], [Woodhouse], [Peskin-Schroeder], [Ramond], [Zee], [Gawedzki-1999], [Teichner]

17:00-18:00 **Question Session with Exercises**

Wednesday, March 1, 2006

9:30-10:30 (4) Spin structures.

- Definition of spin structures via structure group (double cover of SO_n)
- Definition of Clifford algebras, $Spin_n \subset Cl_n$.
- Definition of Spin structures via Clifford bi-modules; opposite spin structures
- Spin structures on manifolds, existence of spin structures (Stiefel-Whitney classes), spin diffeomorphisms. Examples: spin structures on S^1 and on the torus.
- Spinor bundle.

Literature: [Stolz-Teichner], [Lawson-Michelsohn]

11:00-12:00 (5) Dirac operators.

- Definition of Dirac operator on spinor bundle. Connection on spinor bundle.
- The conformal case. Dirac operator on weighted spinor bundle.
- Twisted Dirac operators.
- Index of Dirac operator on spin manifold.

Literature: [Stolz-Teichner], [Hirzebruch-Berger-Jung], [Lawson-Michelsohn], [Bismut-Freed].

14:00-15:00 (6) Fockspaces I.

- Pfaffian line associated to Dirac on closed surfaces
- Idea for ”Generalized Pfaffian”: Fockspace (on surfaces with boundary).
- Input datum for Fockspace construction: Lagrangians. Definition
- Construction of the Fockspace module, creation and annihilation operators
- Segal-Shale equivalence criterion
- Spaces of Lagrangians, orientations and bimodules

Literature: [Stolz-Teichner].

15:30-16:30 (7) Fockspaces II.

- Functorial aspects of the construction
- Generalized Lagrangians and Fockspaces
- The algebras and Fockspaces associated to spin manifolds. On closed surfaces: Pfaffian.
- Gluing of Fockspaces. Vacuum vectors.
- Example: Fockspaces associated to annuli. Fockspace-line bundle over upper half plane.

Literature: [Stolz-Teichner].

17:00-18:00 Question Session with Exercises

Thursday, March 2, 2006

9:30-10:30 (8) **Supermanifolds.**

- Definition of supermanifolds, morphisms in the category.
- The functor $Vect \to \mathcal{SM}$. Examples of supermorphisms which are not bundle morphisms
- Some super geometry: super tangent bundles, metric structures on supermanifolds.
- $\mathbb{R}^{1|1}$ with metric structure. Translations on $\mathbb{R}^{1|1}$ (preserve standard metric structure). The super intervals as sub-supermanifolds.
- Super groups. $\mathbb{R}^{1|1}$ as super (Lie) group.
- The super space of super intervals.

Literature: [Deligne-Morgan], [Varadarajan], [Freed].

11:00-12:00 (9) **The definition of 1-dimensional EFT's a la Stolz/Teichner.**

- The category of super bordisms. Composition. The super moduli space of super bordisms.
- The category of graded Hilbert spaces; the Hilbert universe.
- The degree datum: Fockspace elements associated to bordisms, linearity of the functors; Clifford structure on Hilbert spaces and operators.
- The standard example of a supersymmetric EFT; compare with non-susy standard example.

Literature: [Stolz-Teichner], [Markert].

12:15-13:15 **Question Session with Exercises**

Free Afternoon

Friday, March 3, 2006

9:30-10:30 (10) **From euclidean field theories to K-theory I.**

- Theorem: Spaces of susy EFT's of degree n form spectrum for KO-theory.
- Generating operators of EFT's: bijective correspondence. Configurations.
- Topology on the spaces of generators/configurations.
- The effect of supersymmetry: Contractibility of spaces of non-susy EFT's, comparison of susy and non-susy generators/configurations.

Literature: [Stolz-Teichner], [Markert], [Teichner].

11:00-12:00 (11) **Fredholm Operators and K-theory.**

- (Clifford-linear) Fredholm operators; relation with compact operators
- The Fredholm operator spectrum for K-theory
- Outline of proof of the above theorem

Literature: [Teichner], [Atiyah-Singer].

14:00-15:00 (12) **From euclidean Field Theories to K-theory II.**

- Proof of above theorem
- The partition function of (supersymmetric) euclidean field theories.
- The family index

Literature: [Stolz-Teichner], [Teichner], [Atiyah-Singer].

15:30-16:30 (13) **Overview: the two-dimensional case.**
(E. Markert, G. Laures)

17:00-18:00 **Question Session with Exercises**

Saturday, March 4, 2006

9:30-10:30 (14) **Von Neumann algebras and their bimodules.**

- Von Neumann algebras
- Bimodules over von Neumann algebras
- The Connes Fusion product
- Factors

Literature: [Stolz-Teichner], [Wassserman], [Bratelli-Robinson].

11:00-12:00 (15) **The partition function of CFT's: Connection to modular forms.**

- Reminder partition function of EFT's
- Partition function of a 2-dimensional conformal field theory: Definition
- Modular forms, q-expansion, weak and strong modular forms, the ring of modular forms.
- Partition function of CFT is (weak) modular form.

Literature: [Stolz-Teichner, 3.3], [Segal], [Hirzebruch-Berger-Jung]

12:00 **End**

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