Working and discussion seminar

Problems and recent developments in hyperk geometry

Summer term 2019
Tuesday 2:15– 3:45, Room 0.011

The format for the seminar would be somewhat different from the usual one. The talks will only be loosely connected by the topic 'hyperkähler', which will be broadly understood. Of course, K3 surfaces, even in positive characteristic, and cubic fourfolds count.

Everyone can decide for him- or herself what to talk about, maybe after consultation with some other participants to see whether the topic would be of interest, as long as one of the following criteria applies (ideally, more than one):

(i) A recent new theorem or new technique in hyperkähler geometry is presented.

(ii) The state of the art concerning an open problem or a set of questions is explained.

(iii) You are working on a problem and you would like to present your progress.

Ideally, every talk should lead to a discussion afterwards. We will arrange for tea after the seminar so that there is a well-defined slot for the discussion. You should be willing to share your insight and maybe embark on a joint project. Every talk should contain a set of problems and conjectures. Maybe those could be collected in an official problem list in hyperkähler geometry.

The first topics have been decided on.

1. Lagrangian fibrations I Review of Matshushita's work [18, 19, 20, 21, 22, 23]. The base of a Lagrangian fibration is conjectured to be $\mathbb{P}^n$. This has been proved for smooth base and in dimension four [27, 3, 9]. The work of Hwang et al [10, 11, 12] will only be mentioned in passing. We shall also have a look at [13]. The existence of Lagrangian fibration on hyperkähler manifolds of K3$^{[n]}$-type is another topic.

2. Lagrangian fibrations II The recent paper [29] introduces new techniques and proves a conjecture that is open for Hitchin systems. It also generalizes work of Oguiso [26].

3. Derived categories of hyperkähler manifolds The group of auto-equivalences of the derived category $D(\mathrm{Coh}(S))$ of a K3 surface is described by Brigdeland's conjecture, which is only partially proved. The picture in higher-dimensions is unclear. The existing results construct specific examples, but no general structure results seem known. The aim of the talk should be to give a survey of the existing constructions and ideally formulate some general questions. The following list is probably not complete: [1, 8, 28, 15]. Other natural questions include how to decide when two hyperkähler

manifolds (for example, Hilbert schemes or generalized Kummer varieties) are derived equivalent[25].

4. Twistor lines of supersingular K3/HKs For supersingular K3 surfaces and hyperkähler varieties the notion of twistor lines has recently been introduced [4, 5, 16]. Roughly, it starts with a family of Brauer classes parametrized by $\mathbb{A}^1$ and then, by taking a family of moduli spaces, one does indeed obtain a family of K3 surfaces or higher-dimensional hyperkähler manifolds. How appropriate is the name 'twistor lines' when compared to the twistor lines in complex geometry? What is the use of twistor lines for supersingular K3 surfaces beyond Tate conjecture and unirationality?

5. Monodromy The monodromy of hyperkähler manifolds deformation equivalent to Hilbert schemes of K3 surfaces is well understood [17]. On the other hand, the cohomology of Hilbert schemes has been described in terms of Nakajima operators. This talk will introduce both aspects and show they are related.

6. Automorphisms There has been a stream of articles on symplectic and non-symplectic automorphisms of Hilbert schemes of K3 surfaces and their deformations [6, 7, 24]. The aim of the talk should be to sketch the overall picture. What are the general results or expectations (for example, in comparison with the situation of K3 surfaces)? How do the existing examples fit into the picture? What are the standard methods to construct automorphisms? How do automorphisms fit into the larger framework of auto-equivalences?

7. Fano visitors It has been asked (or even conjectured) that for any K3 surface S or, more generally, any hyperkähler manifold whether there exists a Fano variety X such that $D^b(S)$ can be realized as a full subcategory of $D^b(X)$. For example, [14] proves the existence of such a Fano variety for all complete intersections (not inly K3s!). Other (non-K3) examples include derived categories of curves as subcategories of moduli spaces of bundles on them. For K3 surfaces of certain degrees the Fano can be taken to be a cubic fourfold (Kuznetsov) and from there one could consider Hilbert schemes of the K3 surfaces, etc. The talk should give a overview of the existing examples and techniques.

8. Hodge numbers of hyperkähler manifolds Special guest: Giulia Saccà (Columbia University).

9. Constant cycle subvarieties Constant cycle curves in K3 surfaces are curves that realize only one class in the Chow group (the Beauville–Voisin class). There are open question for K3 surfaces, but more realistic problems exist in higher dimensions, for example existence of constant cycle Lagrangians, etc. It would be good to collect the known results, explain what is expected and propose new situations where geometric construction could be interesting to look at.

List of dates and speakers:

April 9: Daniel Huybrechts: Lagrangian fibrations I

April 16: No seminar (?). Schiermonnikoog

April 23: Isabell Grosse-Brauckmann: Lagrangian fibrations II

April 30: Gebhard Martin: Twistor lines I

May 7: Gebhard Martin: Twistor lines II

May 14: Thorsten Beckmann: Derived categories

May 21: No seminar. Felix-Klein lectures.

May 28: Alberto Cattaneo: Automorphisms

June 4: Georg Oberdieck: Monodromy vs Nakajima

June 11: Break and SFB conference.

June 18: Pieter Belmans: Fano visitors

June 25:?? TBC

July 2: Giulia Saccà (Columbia University): Betti and Hodge numbers of exceptional hyperkähler manifolds

July 9: ?? TBC

Further ideas of possible topics

- Global Torelli for singular hyperkähler varieties (Bakker–Lehn)

- Stability conditions on Kuznetsov's component

- Gromov–Witten methods in hyperkähler geometry

- Cohomology of Betti numbers (known restrictions, using integrable system)

- Franchetta conjecture

- Aspects of moduli space theory (work of Bergeron and Li)

- ...

If you wish to participate and/or suggest a talk, please send an email to huybrech@.....

References

[1] N. Addington *New derived symmetries of some hyperkähler varieties,* arXiv:1112.0487

[2] S. Boissiere, A. Cattaneo, D. Markushevich, A. Sarti *On the nonsymplectic involutions of the Hilbert square of a K3 surface,* arXiv:1805.10481

[3] F. Bogomolov, N. Kurnosov *Lagrangian fibrations for IHS fourfold,* arXiv:1810.11011

[4] D. Bragg *Derived equivalences of twisted supersingular K3 surfaces,* arXiv:1811.07379

[5] D. Bragg, M. Lieblich *Twistor spaces for supersingular K3 surfaces,* arXiv:1804.07282 als `https://digital.lib.washington.edu/researchworks/bitstream/handle/1773/42456/Bragg_washington_0250E_18884.pdf?sequence=1&isAllowed=y`

[6] C. Camere, A. Cattaneo, A. Cattaneo *Non-symplectic involutions on manifolds of $K3^{[n]}$-type,* arXiv:1902.05397

[7] A. Cattaneo *Automorphisms of Hilbert schemes of points on a generic projective K3 surface,* arXiv:1801.05682

[8] D. Huybrechts, R. Thomas $\mathbb{P}$-*objects and autoequivalences of derived categories,* arXiv:math/0507040

[9] D. Huybrechts, C. Xu *Lagrangian fibrations of hyperkhler fourfolds,* arXiv:1902.10440

[10] J.-M. Hwang, K. Oguiso *Multiple fibers of holomorphic Lagrangian fibrations,* arXiv:0907.4869

[11] J.-M. Hwang, K. Oguiso *Local structure of principally polarized stable Lagrangian fibrations,* arXiv:1007.2043

[12] J.-M. Hwang *Base manifolds for fibrations of projective irreducible symplectic manifolds,* arXiv:0711.3224

[13] L. Kamenova, M. Verbitsky *Pullbacks of hyperplane sections for Lagrangian fibrations are primitive,* arXiv:1612.07378

[14] Y.-H. Kiem, I.-K. Kim, H. Lee, K.-S. Lee *All complete intersection varieties are Fano visitors,* arXiv:1503.00125

[15] A. Krug *On derived autoequivalences of Hilbert schemes and generalized Kummer varieties,* arXiv:1404.2105

[16] C. Liedtke *Supersingular K3 surfaces are unirational,* arXiv:1304.5623

[17] E. Markman *Markman, A survey of Torelli and monodromy results for holomorphic-symplectic varieties,* arXiv:1101.4606

[18] D. Matsushita *On fibre space structures of a projective irreducible symplectic manifold,* arXiv:alg-geom/9709033

[19] D. Matsushita *Addendum to: On fibre space structures of a projective irreducible symplectic manifold,* arXiv:math/9903045

[20] D. Matsushita *Equidimensionality of complex Lagrangian fibrations,* arXiv:math/9911166

[21] D. Matsushita *Higher direct images of Lagrangian fibrations,* arXiv:math/0010283

[22] D. Matsushita *A canonical bundle formular of projective Lagrangian fibrations,* arXiv:0710.0122

[23] D. Matsushita *On deformations of Lagrangian fibrations,* arXiv:0903.2098

[24] G. Mongardi *On natural deformations of symplectic automorphisms of manifolds of* $K3^{[n]}$*-type,* arXiv:1304.6630

[25] Y. Namikawa *Counter-example to global Torelli problem for irreducible symplectic manifolds,*arXiv:math/0110114

[26] K. Oguiso *Picard number of the generic fiber of an abelian fibered hyperkähler manifold,* arXiv:0803.1205

[27] W. Ou *Lagrangian fibrations on symplectic fourfolds,* arXiv:1411.1377

[28] D. Ploog, P. Sosna *On autoequivalences of some Calabi–Yau and hyperkähler varieties,* arXiv:1212.4604

[29] J. Shen, Q. Yin *Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds,* arXiv:1812.10673

4