SUPERCUSPIDAL L-PACKETS VIA ISOCRYSTALS

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Abstract

In a recent paper [DR09], DeBacker and Reeder construct and parameterize L-packets on pure inner forms of unramified p-adic groups, that consist of depth zero supercuspidal representations. We generalize their work to non-pure inner forms, by providing an alternative construction based on the theory of isocrystals with additional structure due to Kottwitz. Furthermore, we show the stability and endoscopic transfer for these L-packets.

Given a reductive group G over a local field F, the local Langlands correspondence seeks to parameterize the irreducible admissible representations of $G(F)$ by means of Langlands parameters. Each Langlands parameter is supposed to correspond to an L-packet – a finite set of representations of $G(F)$ that satisfies many properties. Recently, DeBacker and Reeder have considered groups G which are pure inner forms of unramified groups, and a certain class of elliptic Langlands parameters. To each such parameter they construct a finite set of depth zero supercuspidal representations of $G(F)$ and its pure inner forms. The stability of these packets was proved in [DR09], and their endoscopic transfer was proved in [Kal09].

The purpose of this paper is to generalize this work to non-pure inner forms of unramified groups. The concept of pure inner forms was introduced by Vogan, who realized that the notion of an inner form is not rigid enough for the purposes of the local Langlands correspondence. Unfortunately, not all inner forms can be rigidified to pure inner forms. Kottwitz observed that isocrystals with additional structure can be used to provide a rigidification of inner forms of p-adic groups that fits very naturally in the framework of the local Langlands correspondence and endoscopy, and moreover has the potential to rigidify all inner forms of a given quasi-split group. We use Kottwitz's idea and his work [Kot85], [Kot97] on isocrystals to give a construction of L-packets on these rigidified inner forms. Our construction is different from that of DeBacker and Reeder, but gives rise to the same packets on pure inner forms. Moreover, we prove the stability and endoscopic transfer of the depth-zero packets. In the case of a connected center, all inner forms can be rigidified using isocrystals. The case of a disconnected center can be reduced to that of a connected center. The techniques and difficulties involved in this reduction are of a separate nature and will be pursued in a forthcoming paper.

To explain the results more precisely, let F be a p-adic field with Weil group W_F and G an unramified connected reductive group defined over F. In [Kot97], Kottwitz introduces a set $\mathbf{B}(G)_b$, which consists of certain cohomology classes, and shows how each element $b \in \mathbf{B}(G)_b$ gives rise to an equivalence class of inner forms G^b of G. Let $\widehat{G}$ be a Langlands dual group for G, and $^L G = \widehat{G} \rtimes W_F$ the corresponding L-group. The first main result of this paper is the explicit construction and parametrization of a packet $\Pi^b_{[\varphi]}$ of representations of $G^b(F)$ for each $b \in \mathbf{B}(G)_b$ and each Langlands parameter

$$\varphi : W_F \to {}^L G$$

which is subject to certain properties (see Section 3.2). This packet depends only on the $\widehat{G}$-conjugacy class $[\varphi]$ of φ and on no auxiliary choices.

The packet $\Pi_{[\varphi]}^b$ is parameterized in the following way: Consider the group $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$. The properties of φ imply that this is a diagonalizable subgroup of $\widehat{G}$ and contains $Z(\widehat{G})^\Gamma$ as a subgroup of finite index. It depends only on $[\varphi]$ (see the beginning of Section 3.3 for a discussion), and we write $S_{[\varphi]}$ for it. For each hyperspecial vertex o in the reduced building of $G(F)$, we construct a bijection

$$X^*(S_{[\varphi]}) \to \bigsqcup_{b \in \mathbf{B}(G)_b} \Pi_{[\varphi]}^b. \tag{0.1}$$

We show that this bijection satisfies the following two properties: First, it maps the trivial character to a representation of $G(F)$ which is generic with respect to a character of generic depth 0 at o (see [DR08] for terminology). Second, it identifies each set $\Pi_{[\varphi]}^b$ with the fiber over b of the map

$$X^*(S_{[\varphi]}) \twoheadrightarrow X^*(Z(\widehat{G})^\Gamma) \to \mathbf{B}(G)_b,$$
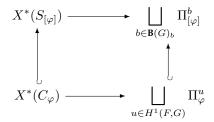
where the first map is the natural restriction and the second is the isomorphism constructed by Kottwitz.

We give an explicit formula for the dependence of the bijection (0.1) on the choice of the hyperspecial vertex o. More precisely, for each pair of hyperspecial vertices o, o' we construct an element $(o, o') \in X^*(S_{[\varphi]})$ and show that the two versions of (0.1) corresponding to o and o' differ by translation by (o, o').

Our second main result is the compatibility of (0.1) with the construction of [DR09]. The Langlands parameters we are considering are the same as the ones considered there. In that paper, the authors construct for each pure inner form G^u of G, where $u \in H^1(F, G)$, a packet Π_φ^u on $G^u(F)$ and a bijection

$$X^*(C_\varphi) \to \bigsqcup_{u \in H^1(F,G)} \Pi_\varphi^u,$$

where $C_\varphi = \pi_0(S_\varphi)$. There is a natural injection $H^1(F, G) \to \mathbf{B}(G)_b$ and if b is the image of u, then $G^b = G^u$. We show that in this case, we have $\Pi_{[\varphi]}^b = \Pi_\varphi^u$, and moreover we have a commutative diagram

$$
\begin{array}{ccc}
X^*(S_{[\varphi]}) & \longrightarrow & \bigsqcup_{b \in \mathbf{B}(G)_b} \Pi_{[\varphi]}^b \\
\uparrow & & \uparrow \\
X^*(C_\varphi) & \longrightarrow & \bigsqcup_{u \in H^1(F,G)} \Pi_\varphi^u
\end{array}
$$

In this sense, our work is an extension of theirs. While our construction uses different methods and in particular does not rely on the combinatorial arguments in [DR09] involving the Bruhat-Tits building, the reader will clearly notice the influence of this paper.

The remaining results concern character identities and rely on the character computations in [DR09, §§8,9,10,12]. These computations are derived for general p-adic groups which split over an unramified extension of F, and can be used in our setting. For each $b \in \mathbf{B}(G)_b$ and $s \in S_{[\varphi]}$ we define

$$\Theta_{[\varphi],b}^s = e(G^b) \sum_{\substack{\rho \in X^*(S_{[\varphi]}) \\ \rho \mapsto b}} \rho(s) \Theta_{\pi_\rho},$$

2

where $\rho \mapsto \pi_\rho$ is the map (0.1), Θ_{π_ρ} is the character of the representation π_ρ, which we view as a function on the set of strongly regular semi-simple elements of $G^b(F)$, and $e(G^b)$ is the Kottwitz sign of G^b. Our third main result is the stability of the family

$$\left\{ \Theta^1_{[\varphi],b} \right\}_{b \in \mathbf{B}(G)_b}$$

in the following sense: There is a notion of when two semi-simple elements $\gamma^b \in G^b(F)$ and $\gamma^c \in G^c(F)$ for $b, c \in \mathbf{B}(G)_b$ are stably conjugate (see Section 2), and we show that if this is the case, then

$$\Theta^1_{[\varphi],b}(\gamma^b) = \Theta^1_{[\varphi],c}(\gamma^c).$$

Our final result is the validity of endoscopic induction for our L-packets. Let $(H, s, {}^L\eta)$ be an unramified extended endoscopic triple for G, $[\varphi^H]$ a Langlands parameter for H, and $[\varphi] = [{}^L\eta] \circ [\varphi^H]$. Associated to the bijection (0.1) there is a compatible family of normalizations $\Delta^{G^b}_H$ of the transfer factors for all pairs (G^b, H) with $b \in \mathbf{B}(G)_b$. We define the endoscopic lift to $G^b(F)$ of the stable character corresponding to $[\varphi^H]$ by

$$\mathrm{Lift}^{G^b}_H \Theta^1_{[\varphi^H],0}(\gamma^b) := \sum_{\gamma^H} \Delta^{G^b}_H(\gamma^H, \gamma^b) \frac{D^H(\gamma^H)^2}{D^{G^b}(\gamma^b)^2} \Theta^1_{[\varphi^H],0}(\gamma^H),$$

where γ^H runs over the stably conjugacy classes of strongly regular semi-simple elements of $H(F)$ and D are the usual Weyl discriminants. We show that

$$\mathrm{Lift}^{G^b}_H \Theta^1_{[\varphi^H],0} = \Theta^s_{[\varphi],b}.$$

We will now briefly describe the contents of this paper. In section 2 we lay out Kottwitz's ideas on using isocrystals to rigidify inner forms and obtain good notions of rational and stable conjugacy of elements lying in different inner forms, as well as compatible normalizations of transfer factors across inner forms. We call the rigidified inner forms extended pure inner forms, or ep forms for short. The set $\mathbf{B}(G)_b$ parameterizes the equivalence classes of ep forms. We want to be able to work with actual ep forms, rather than just equivalence classes, and we introduce for that purpose the set $E(G)$. It is the analog of the set $Z^1(F, G)$ of pure inner forms. Section 2 is valid for any connected reductive p-adic group. Section 3 is devoted to the L-packets. In 3.1 we review the representations that constitute our packets. These representations are very well known and we limit ourselves to just gathering the properties that will be important for us. For an exposition on their construction, we refer the reader to [DR09, §4.4]. In 3.2 we introduce the Langlands parameters for our packets. They are the same as the ones considered in [DR09]. The actual construction of the packets and their parameterization is carried out in 3.3. In order to construct the packet for a given parameter, we first construct a triple $(S_0, [a], [{}^L j])$, and for that we make some auxiliary choices. In 3.4 we show that the properties of the triple $(S_0, [a], [{}^L j])$ make it essentially unique, so none of the choices we made actually matters, thereby showing the canonicity of our construction. Moreover, we give in 3.4 the formula for the dependence of the bijection (0.1) on the choice of the hyperspecial vertex o. In 3.5 we prove the compatibility of our packets with those constructed by DeBacker and Reeder. Finally 3.6 provides some facts needed for the proper normalization of the transfer factors. Section 4 deals with stability and endoscopic transfer. After establishing a reduction formula for the unstable character in 4.2, we show stability and

endoscopic transfer in 4.3. In fact, the endoscopic transfer follows from the arguments of [Kal09], after a few key statements have been generalized to our situation. Most notably, these statements include the reduction formula from 4.2, and the sign computation from 4.1. After that, the argument of [Kal09, §7] goes through verbatim, and we refer the reader to it. While the stability of our packets is a special case of their endoscopic transfer, the argument for this special case is significantly simpler and shorter, and so we give a direct proof for it.

In order to derive our results, we have to impose some restrictions on the field F. For the construction of our packets in section 3, we assume that p is odd. In section 4, we impose the same restrictions on F as in [DR09, §12] and [Kal09, §3]. Moreover, we require in this section that the center of G be connected. This last requirement is not strictly necessary, but considerably simplifies the proof of Proposition 2.2.1. The validity of this proposition in general can be deduced via reduction to the case of a connected center, which will be done in the forthcoming paper mentioned above.

We would like to mention that the constructions in Section 3 are not limited to the depth-zero case. They apply equally well in situations of positive depth, and we can show that they provide a generalization to non-pure inner forms of the packets constructed in [Ree08] as well. We will address this in a future paper, once the stability of these packets, which is a work in progress by DeBacker and Spice, has been established.

CONTENTS

1   NOTATION AND CONVENTIONS

Throughout this paper, F denotes a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$, with ring of integers O_F and residue field k_F of cardinality q_F. We fix an algebraic closure $\overline{F}$ of F, and let Γ denote the absolute Galois group of F, W_F the Weil group, and I_F the inertia subgroup. We fix a uniformizer $\pi \in O_F$, as well as an element $\Phi \in \Gamma$ whose inverse induces the map $x \mapsto x^{q_F}$ on $\overline{k_F}$. Let F^u be the maximal unramified extension of F in $\overline{F}$, L the completion of F^u, and $\overline{L}$ a fixed algebraic closure of L, which we take to contain $\overline{F}$. Recall that $\overline{L} \cong L \otimes_{F^u} \overline{F}$. The diagonal action of Γ on the right hand side of this isomorphism is well-defined and in this way we obtain a continuous action of Γ on $\overline{L}$ whose fixed field is F.

If G is a connected reductive group defined over F, we will use the Fraktur letter $\mathfrak{g}$ for its Lie-algebra. If $T \subset G$ is a maximal torus (by which we mean that it is defined over F), we write $R(T, G)$ for the set of roots of T in G, $N(T, G)$ for the normalizer and $\Omega(T, G)$ for the Weyl group of T in G. If $B \subset G$ is a Borel containing T (again defined over F), then $R(T, B)$ is a choice of positive roots inside $R(T, G)$ and we write $\Delta(T, B)$ for the corresponding set of simple roots. In this case, T is a minimal Levi of G, and we will call it also a quasi-split

maximal torus of G. If T_1, T_2 are two maximal tori in G and $\mathrm{Ad}(h) : T_1 \to T_2$ with $h \in G(\overline{F})$ is an isomorphism, then the dual of this isomorphism will be denoted by $\widehat{\mathrm{Ad}}(h) : \widehat{T_2} \to \widehat{T_1}$. The subset of strongly regular semi-simple elements of G will be denoted by G_{sr}. We will write $Z^1(F, G)$ or $Z^1(\Gamma, G)$ for the set of continuous 1-cocycles of Γ in the discrete group $G(\overline{F})$, and we will write $Z^1(W_F, G(\overline{L}))$ for the set of continuous 1-cocycles of W_F in the discrete group $G(\overline{L})$. In [Kot97], Kottwitz defines a subset $Z^1(W_F, G(\overline{L}))_b$ of "basic" 1-cocycles. If $u \in Z^1(F, G)$ is an unramified cocycle, we will abuse notation and use the letter u also for the value of u at Φ. Given an element $g \in G$, we will write $\mathrm{Cent}(g, G)$ or G^g for the centralizer of g in G, and G_g for its connected component.

We will write $\mathcal{B}^{\mathrm{red}}(G, F)$ for the reduced building of $G(F)$, and $\mathcal{A}^{\mathrm{red}}(T, F)$ for the reduced apartment of any maximal split torus $T \subset G$. For any $x \in \mathcal{B}^{\mathrm{red}}(G, F)$, we let $G(F)_x$ be the fixator of x for the action of $G(F)$ on $\mathcal{B}^{\mathrm{red}}(G, F)$, and $G(F)_{x,0}$ resp. $G(F)_{x,0+}$ be the parahoric subgroup corresponding to x resp. its unipotent radical. If T is an unramified maximal torus of G, then it is a maximal split torus in $G \times F^u$, and hence we have the apartment $\mathcal{A}^{\mathrm{red}}(T, F^u)$ inside $\mathcal{B}^{\mathrm{red}}(G, F^u)$. In this situation, we will denote $\mathcal{A}^{\mathrm{red}}(T, F^u) \cap \mathcal{B}^{\mathrm{red}}(G, F)$ by $\mathcal{A}^{\mathrm{red}}(T, F)$, even though this will in general not be an apartment in $\mathcal{B}^{\mathrm{red}}(G, F)$.

If A is a finite group, we write $A^D = \mathrm{Hom}(A, \mathbb{C}^\times)$. More generally if A is a complex diagonalizable group, we will write $A^D = X^*(A)$.

2   EXTENDED PURE INNER TWISTS

2.1   Definition and basic properties

We recall some basic facts about isocrystals with additional structure and fix some more notation to be used throughout the rest of the paper. Let A be a connected reductive group defined over F. To any element $z \in Z^1(W_F, A(\overline{L}))$ one can define [Kot97, §3.2] a homomorphism of L-groups

$$\nu_z : \mathbb{D} \to A,$$

where $\mathbb{D}$ is the pro-diagonalizable group whose character group is the trivial Γ-module $\mathbb{Q}$. The conjugacy class of ν_z depends only on the cohomology class of z. We can use ν_z to define two distinguished subsets of

$$\mathbf{B}(A) = H^1(W_F, A(\overline{L})).$$

First, we can consider the set of those classes $[z]$ whose corresponding maps ν_z are trivial. It is shown in [Kot97, §3.2] that this is precisely the image of the natural injection

$$H^1(F, A(\overline{F})) \to \mathbf{B}(A).$$

Next, we can consider the set of all $z \in Z^1(W_F, A(\overline{L}))$ for which ν_z factors through $Z(A)$. Those z are called *basic*, their set is denoted by $Z^1(W_F, A(\overline{L}))_b$ and the set of their cohomology classes by $\mathbf{B}(A)_b$.

Kottwitz has shown that each basic element $z \in Z^1(W_F, A(\overline{L}))$ gives rise to an inner form of A. We'd like to recall this process in a slightly different form. Let

$E(A)$ be the following pull-back

$$
\begin{array}{ccc}
E(A) & \xrightarrow{\;p_2\;} & Z^1(W_F, A(\overline{L}))_b \\[4pt]
\Big\downarrow{\scriptstyle p_1} & & \Big\downarrow \\[4pt]
Z^1(\Gamma, A_{\mathrm{ad}}(\overline{F})) & \lhook\joinrel\longrightarrow & Z^1(W_F, A_{\mathrm{ad}}(\overline{L}))_b
\end{array}
\tag{2.1}
$$

An extended pure inner twist (ep twist) $(\psi, b) : A \to B$ consists of a a connected reductive group B defined over F, an isomorphism $\psi : A \to B$ of $\overline{F}$-groups and an element $b \in E(A)$ such that

$$
\psi^{-1}\sigma(\psi) = \mathrm{Ad}(p_1 b(\sigma)) \qquad \forall \sigma \in \Gamma.
$$

If $(\psi, b) : A \to B$ and $(\varphi, c) : B \to C$ are ep twists, then we can form

$$
(\psi, b)^{-1} : B \to A, \qquad (\varphi, c) \circ (\psi, b) : A \to C
$$

by $(\psi, b)^{-1} = (\psi^{-1}, \psi(b)^{-1})$ and $(\varphi, c) \circ (\psi, b) = (\varphi \circ \psi, \psi^{-1}(c)b)$ where multiplication and inversion of elements of E is to be taken component- and pointwise. Two ep twists $(\psi_i, b_i) : A \to B$ are called equivalent if there exists an element $(x, y) \in A_{\mathrm{ad}}(\overline{F}) \times_{A_{\mathrm{ad}}(\overline{L})} A(\overline{L})$ such that

$$
(\psi_1, b_1)^{-1} \circ (\psi_2, b_2) = (\mathrm{Ad}(x), (x^{-1}\sigma(x), y^{-1}\sigma(y))).
$$

The twist (φ, b) is called strongly trivial if $b = 1$, and trivial if it is equivalent to a strongly trivial twist.

Fact 2.1.1.

1. *The map $E(A) \to B(A)_b$ induced by p_2 is surjective.*

2. *If $Z(A)$ is connected, then the map $E(A) \to H^1(F, A_{\mathrm{ad}})$ induced by p_1 is surjective.*

Proof. We recall the well-known arguments for the convenience of the reader. Let $[z] \in B(A)_b$. Using Steinberg's theorem we may choose z to be an unramified cocycle $\langle \Phi \rangle \to A(L)$. Using [Kot85, (4.3.3)] we may further assume that the value of z at Φ^n, for some suitable integer n, belongs to $Z(A)$. Thus the prolongation of z to A_{ad} is trivial on the subgroup of $\langle \Phi \rangle$ generated by Φ^n, which shows that z belongs to $E(A)$.

Assuming now that $Z(A)$ is connected, we need to show that in the diagram

$$
H^1(\Gamma, A_{\mathrm{ad}}(\overline{F})) \to \mathbf{B}(A_{\mathrm{ad}})_b \leftarrow \mathbf{B}(A)_b
$$

the image of the first map is contained in the image of the second map. The first map is a bijection because $Z(A_{\mathrm{ad}}) = \{1\}$, and the second map is surjective because it is dual to the map $Z(\widehat{G}_{\mathrm{sc}})^\Gamma \to Z(\widehat{G})^\Gamma$ [Kot97, §4], which is injective due to our assumption. $\qquad\square$

We can now redo all definitions and arguments of [Kal09, §2.1] with the functor $A \mapsto H^1(\Gamma, A(\overline{F}))$ replaced by $A \mapsto \mathbf{B}(A)_b$. It is straightforward to see that all statements stated there for pure inner twists remain valid for ep twists: The statement of Fact 2.1.1(1) follows from the next lemma, the statement of Fact 2.1.2(2) is to be interpreted to mean that the new and old definitions of inv correspond under the injection $H^1(\Gamma, A_a(\overline{F})) \to \mathbf{B}(A_a)_b$, and the other statements remain unchanged.

Lemma 2.1.2. *Let $a, b \in A(F)$, and let a_s be the semi-simple part of a. Put*

$$
\begin{aligned}
C(a, b; \overline{L}) &= \{g \in A(\overline{L}) | \mathrm{Ad}(g)a = b \wedge \forall \sigma \in \Gamma : g^{-1}\sigma(g) \in G_{a_s}(\overline{L})\}, \\
C(a, b; \overline{F}) &= \{g \in A(\overline{F}) | \mathrm{Ad}(g)a = b \wedge \forall \sigma \in \Gamma : g^{-1}\sigma(g) \in G_{a_s}(\overline{F})\}.
\end{aligned}
$$

Then $C(a, b; \overline{L}) \neq \emptyset \Leftrightarrow C(a, b; \overline{F}) \neq \emptyset$.

Proof: For the duration of this proof, we will say that a, b are $\overline{L}$-stably conjugate if the first set is non-empty, and $\overline{F}$-stably conjugate if the second set is non-empty.

Assume first that A has a simply-connected derived group. Then $G^a \subset G^{a_s} = G_{a_s}$ and the second conditions in the definitions of the above sets are vacuous. The sets themselves are then the sets of $\overline{L}$-points resp. the $\overline{F}$-points of the algebraic variety $\{g \in A | \mathrm{Ad}(g)a = b\}$. This variety is defined over F and the statement follows.

We reduce the general case to this by taking a z-extension $A' \to A$. The argument of the proof of [Kot82, 3.1.(2)] shows that a, b are $\overline{L}$-stably conjugate if and only if there exist lifts $a', b' \in A'(F)$ which are $\overline{L}$-stably conjugate. Applying the proved special case this is equivalent to a', b' being $\overline{F}$-stably conjugate, which again by loc. cit. is equivalent to a, b being $\overline{F}$-stably conjugate. $\qquad\square$

By a representation of an ep twist of A we shall mean a quadruple (B, ψ, b, π) where $(\psi, b) : A \to B$ is an ep twist and π is a representation of $B(F)$. Two such quadruples (B, ψ, b, π) and (B', ψ', b', π') will be called equivalent if there exists a strongly-trivial ep twist $B \to B'$ which is equivalent to $\psi'\psi^{-1}$ identifies π with π'. It is easy to check that then every strongly-trivial ep twist $B \to B'$ which is equivalent to $\psi'\psi^{-1}$ and identifies π with π'

2.2 Transfer factors

Let G be a connected reductive group, defined and quasi-split over F, $(\psi, b) : G \to G'$ be a ep twist, and $(H, s, {}^L\eta)$ be an extended endoscopic triple for G. Fix a normalization Δ_G^H of the absolute transfer factor for (G, H). We are going to define a normalization $\Delta_{G'}^H$ of the absolute transfer factor for (G', H) as follows. Let $\gamma' \in G'(F)$ and $\gamma^H \in H(F)$ be a pair of related strongly G-regular elements. Choose any $\gamma \in G(F)$ stably-conjugate to γ', and let $\varphi_{\gamma, \gamma^H} : T_\gamma \to T_{\gamma_H}^H$ be the unique admissible isomorphism from the centralizer of γ in G to the centralizer of γ^H in H. In [Kot85, §2], Kottwitz constructs an isomorphism $X_*(T_\gamma)_\Gamma \to \mathbf{B}(T_\gamma)$, and this isomorphism provides a pairing

$$
\langle \, , \, \rangle : \mathbf{B}(T_\gamma) \times \widehat{T_\gamma}^\Gamma \to \mathbb{C}^\times.
$$

We put

$$
\Delta_{G'}^H(\gamma^H, \gamma') = \Delta_G^H(\gamma^H, \gamma)\langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma, \gamma^H}(s)\rangle^{-1}.
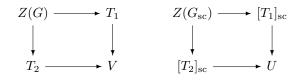$$

The same proof as for [Kal09, Lemma 2.2.1] shows that $\Delta_{G'}^H$ is well-defined. Moreover, using [Kot85, 2.7] and [Kal09, Lemma 2.3.2] one sees that if (ψ, b) is a pure inner twist, then this definition agrees with the one given in [Kal09, §2.2].

Proposition 2.2.1. *Assume that $Z(G)$ is connected. Then $\Delta_{G'}^H$ is an absolute transfer factor for (G', H).*

Proof: For $i = 1, 2$, let $\gamma_i' \in G'(F)$ and $\gamma_i^H \in H(F)$ be a pair of strongly G-regular related elements, and let $\gamma_i \in G(F)$ be stably conjugate to γ_i'. As in the proof of [Kal09, Proposition 2.2.2] one reduces to showing that the following equality holds:

$$\frac{\langle \mathrm{inv}(\gamma_1, \gamma_1'), \widehat{\varphi}_{\gamma_1, \gamma_1^H}(s) \rangle^{-1}}{\langle \mathrm{inv}(\gamma_2, \gamma_2'), \widehat{\varphi}_{\gamma_2, \gamma_2^H}(s) \rangle^{-1}} = \left\langle \mathrm{inv}\left(\frac{\gamma_1, \gamma_1'}{\gamma_2, \gamma_2'} \right), s_U \right\rangle. \tag{2.2}$$

We put $T_i = \mathrm{Cent}(\gamma_i, G)$, $T_i^H = \mathrm{Cent}(\gamma_i^H, H)$, and form the push-out diagrams

$$
\begin{array}{ccc}
Z(G) & \longrightarrow & T_1 \\
\downarrow & & \downarrow \\
T_2 & \longrightarrow & V
\end{array}
\qquad
\begin{array}{ccc}
Z(G_{\mathrm{sc}}) & \longrightarrow & [T_1]_{\mathrm{sc}} \\
\downarrow & & \downarrow \\
[T_2]_{\mathrm{sc}} & \longrightarrow & U
\end{array}
$$

The canonical map

$$[T_1]_{\mathrm{sc}} \times [T_2]_{\mathrm{sc}} \to T_1 \times T_2 \to V$$

factors uniquely through the isogeny $[T_1]_{\mathrm{sc}} \times [T_2]_{\mathrm{sc}} \to U$ to give a map $U \to V$. One sees easily that the image of $\mathrm{inv}(\gamma_1, \gamma_1'/\gamma_2, \gamma_2')$ under the map $H^1(\Gamma, U) \to H^1(\Gamma, V) \to \mathbf{B}(V)$ coincides with the image of $(\mathrm{inv}(\gamma_1, \gamma_1')^{-1}, \mathrm{inv}(\gamma_2, \gamma_2'))$ under the canonical map $\mathbf{B}(T_1) \times \mathbf{B}(T_2) \to \mathbf{B}(V)$. Equation (2.2) will follow from [Kot85, 2.7] and the functoriality of the Tate-Nakayama pairing once we exhibit an element $s_V \in \widehat{V}^\Gamma$, whose image under $\widehat{V} \to \widehat{U}$ equals s_U, and whose image under $\widehat{V} \to \widehat{T}_1 \times \widehat{T}_2$ equals $(\widehat{\varphi}_{\gamma_1, \gamma_1^H}(s), \widehat{\varphi}_{\gamma_2, \gamma_2^H}(s))$.

By Langlands duality we obtain a presentation of $\widehat{V}$ as the pull-back of the dual diagram

$$\widehat{T}_1 \to \widehat{G}/\widehat{G}_{\mathrm{der}} \leftarrow \widehat{T}_2.$$

The image of s under

$$Z(\widehat{H}) \xrightarrow{\quad \Delta \quad} \widehat{T_1^H} \times \widehat{T_2^H} \xrightarrow{\quad \widehat{\varphi}_{\gamma_1, \gamma_1^H}, \widehat{\varphi}_{\gamma_2, \gamma_2^H} \quad} \widehat{T}_1 \times \widehat{T}_2$$

belongs to $\widehat{V}$, call it s_V. Consider the diagrams

$$
\begin{array}{ccc}
[T_1]_{\mathrm{sc}} \times [T_2]_{\mathrm{sc}} & \longrightarrow & T_1 \times T_2 \\
\downarrow & & \downarrow \\
U & \longrightarrow & V
\end{array}
\qquad
\begin{array}{ccc}
\widehat{[T_1]}_{\mathrm{ad}} \times \widehat{[T_2]}_{\mathrm{ad}} & \longleftarrow & \widehat{T}_1 \times \widehat{T}_2 \\
\uparrow & & \uparrow \\
\widehat{U} & \longleftarrow & \widehat{V}
\end{array}
$$

The map $U \to V$ is characterized uniquely by the property that it makes the left diagram commutative, hence its dual is the unique map $\widehat{V} \to \widehat{U}$ making the right diagram commutative. Explicitly, this map is given as follows. Recall from [LS87, §3.4] that $\widehat{U} = \widehat{[T_1]}_{\mathrm{sc}} \times \widehat{[T_2]}_{\mathrm{sc}}/Z(\widehat{G}_{\mathrm{sc}})$ where $Z(\widehat{G}_{\mathrm{sc}})$ is embedded diagonally into $\widehat{[T_1]}_{\mathrm{sc}} \times \widehat{[T_2]}_{\mathrm{sc}}$. Let $(t_1, t_2) \in \widehat{T}_1 \times \widehat{T}_2$ be an element of $\widehat{V}$. Thus t_1 and t_2 have the same image in $\widehat{G}/\widehat{G}_{\mathrm{der}}$. Choose $z \in Z(\widehat{G})$ mapping to that image. Then $(z^{-1}t_1, z^{-1}t_2)$ belongs to $\widehat{[T_1]}_{\mathrm{sc}} \times \widehat{[T_2]}_{\mathrm{sc}}$. The image of $(z^{-1}t_1, z^{-1}t_2)$ in $\widehat{U}$ is independent of the choice of z. The map $\widehat{V} \to \widehat{U}$ obtained in this way obviously makes the right diagram commute and thus must be the dual of $U \to V$. From the explicit description of the map $\widehat{V} \to \widehat{U}$ one reads off that the image of s_V equals s_U, and this concludes the proof of the proposition. $\qquad\square$

Throughout this section, we assume that p is odd.

3.1   The representations

Let G be a connected reductive group defined over F. Consider a pair (T, θ), where $T \subset G$ is an elliptic unramified maximal torus, and θ is a character $\theta : T(F) \to \mathbb{C}^\times$, which is of depth-zero and regular. Recall that a character of $T(F)$ is of depth-zero if its restriction to the pro-p-Sylow subgroup of the maximal compact subgroup of $T(F)$ is trivial, and it is regular if its stabilizer in $\Omega(T, G)$ be trivial.

To such a pair we assign a smooth irreducible supercuspidal representation $\pi_{G,T,\theta}$ of $G(F)$, as was done in [DR09, §4.4]. Let us briefly recall its construction. According to [Deb06, §2.2], $x := \mathcal{A}^{\mathrm{red}}(T, F)$ is a vertex. Let G be the connected reductive group defined over k_F associated to this vertex, and T be the maximal torus in G corresponding to T. The character θ gives rise to a regular character $\theta : \mathsf{T}(k_F) \to \mathbb{C}^\times$. By [DL76, 8.3] one obtains an irreducible cuspidal representation $\epsilon(\mathsf{G}, \mathsf{T}) R_\mathsf{T}^\theta$ of $\mathsf{G}(k_F)$. Let κ^0 denote the inflation of this representation to $G(F)_{x,0}$, and put

$$\pi_{G,T,\theta} = \mathrm{Ind}_{Z(F)G(F)_{x,0}}^{G(F)_x} \left(\theta \otimes \kappa^0 \right).$$

According to [DR09, Lemma 4.5.1], this representation is irreducible and supercuspidal.

Lemma 3.1.1. *The representations π_{G,T_1,θ_1} and π_{G,T_2,θ_2} are isomorphic if and only if the pairs (T_1, θ_1), (T_2, θ_2) are $G(F)$-conjugate*

Proof: Conjugate pairs give rise to conjugate inducing data and hence to isomorphic representations. Conversely assume that $\pi_{G,T_1,\theta_1} \cong \pi_{G,T_2,\theta_2}$. Let $x, y \in \mathcal{B}^{\mathrm{red}}(G, F)$ be the vertices corresponding to T_1, T_2. According to [MP96, 3.5] the depth-zero unrefined minimal K-types $(G(F^u)_{x,0}, \kappa_1^0)$ and $(G(F^u)_{y,0}, \kappa_2^0)$ are associate, hence there exists $g \in G(F)$ such that $gx = y$ and $\mathrm{Ad}(g)\kappa_1^0 = \kappa_2^0$. Thus we may assume $x = y$ and $\kappa_1^0 = \kappa_2^0$. Using [DL76, 6.8] we find $g \in G(F)$ such that $\mathrm{Ad}(g)(\mathsf{T}_1, \theta_1) = (\mathsf{T}_2, \theta_2)$. The lemma now follows from [Deb06, 2.2.2]. $\square$

Lemma 3.1.2. *Let Θ be the character of $\pi_{G,T,\theta}$ as a function on $G_{\mathrm{sr}}(F)$. If $Q_T \in \mathrm{Lie}(T)(F)$ is a fixed regular semi-simple element, then for any $z \in Z_G(F)$ and $\gamma \in G_{\mathrm{sr}}(F)_0$ we have*

$$\Theta(z\gamma) = \epsilon(G, A_G)\theta(z) \sum_Q R(G_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_T,Q}]_* \theta(\gamma_s),$$

where Q runs over any set of representatives for the $G_{\gamma_s}(F)$-conjugacy classes inside the $G(F)$-conjugacy class of Q_T, and $S_Q = \mathrm{Cent}(Q, G)$.

Proof: The proof is the same as for [Kal09, 6.2.1]. $\square$

3.2 The parameters

Let G be an unramified connected reductive group defined over F, $^L G = \widehat{G} \rtimes W_F$ its L-group, and $[\varphi] : W_F \to {}^L G$ an equivalence class of Langlands parameters. If this class satisfies certain conditions, we are going to construct in the next section a triple $(S_0, [a], [^L j])$ where $S_0 \subset G$ is an unramified maximal torus, $[a] : W_F \to {}^L S_0$ is an equivalence class of Langlands parameters, and $[^L j] : {}^L S_0 \to {}^L G$ is an $\widehat{G}$-conjugacy class of unramified L-embeddings such that $[^L j] \circ [a] = [\varphi]$. The parameter $[a]$ gives rise to a character $\theta_0 : S_0(F) \to \mathbb{C}^\times$.

Choose a representative φ within the equivalence class $[\varphi]$, and let $\varphi_0 : W_F \to \widehat{G}$ be the composition of φ with the projection to $\widehat{G}$. Then φ_0 is an element of $Z^1(W_F, \widehat{G})$ whose cohomology class is independent of the choices made. The conditions we are imposing on $[\varphi]$ are the following:

1. φ_0 restricts trivially to the wild inertia subgroup of W_F,

2. the centralizer of $\varphi_0(I_F)$ in $\widehat{G}$ is a maximal torus,

3. the index of $Z(\widehat{G})^\Gamma$ in $\mathrm{Cent}(\varphi, \widehat{G})$ is finite.

These are the Langlands parameters considered in [DR09]. For the construction of the triple $(S_0, [a], [^L j])$, conditions 2 and 3 are enough, thus it is applicable to regular elliptic parameters of positive depth as well. Condition 1 is used in the construction of the representations.

Proposition 3.2.1. *Assume that $[\varphi]$ satisfies conditions 1-3 above, and let θ_0 and $(S_0, [a], [^L j])$ be as before. Then θ_0 is a regular character of depth 0.*

Proof: Let $w \in \Omega(S_0, G)(F)$. By functoriality of the Langlands correspondence we have $w\theta_0 = \theta_0$ if and only if $[a] = [w \circ a]$. If a is any representative of the class of a, then a necessary condition for the second equality is $a|_{I_F} = w \circ a|_{I_F}$ (recall that S_0 is unramified). This however is precluded by condition 2 on φ, and we conclude that θ_0 is regular. Moreover, since the Langlands correspondence preserves depth, θ_0 is of depth 0 (see [Ree08, §4] for an explicit version of the Langlands correspondence for unramified tori). $\qquad\qquad \square$

3.3 The packets

We continue with G, $^L G$ and $[\varphi]$ as in the previous section, and require now that $[\varphi]$ satisfies the properties 1-3 described there. For each $\varphi_1 \in [\varphi]$, let S_{φ_1} be the centralizer of φ_1 in $\widehat{G}$ (note that it is abelian). For $\varphi_1, \varphi_2 \in [\varphi]$ there is a canonical isomorphism $S_{\varphi_1} \to S_{\varphi_2}$, and we define $S_{[\varphi]} = \varinjlim S_{\varphi_1}$, where φ_1 runs over $[\varphi]$. In this section we are going to construct a set $\Pi_{[\varphi]}$ of (equivalence classes of) irreducible admissible representations of ep twists of G and an explicit bijection

$$S^D_{[\varphi]} \to \Pi_{[\varphi]}. \tag{3.1}$$

This bijection is going to depend on a choice of a hyperspecial vertex $o \in \mathcal{B}^{\mathrm{red}}(G, F)$. The left-hand side of the above map is a group and thus has a distinguished element, namely the identity, while the right-hand side is a-priori just a set. The choice of o serves in particular to fix a base-point in this set, and

different choices will lead to different base-points. The set $\Pi_{[\varphi]}$ itself will be independent of the choice of o. Moreover, we will have the diagram

$$
\begin{array}{ccc}
S^D_{[\varphi]} & \longrightarrow & \Pi_{[\varphi]} \\
\downarrow & & \downarrow \\
[Z(\widehat{G})^\Gamma]^D & \longrightarrow & \mathbf{B}(G)_b
\end{array}
$$

where the left vertical map is the dual of the natural inclusion $Z(\widehat{G})^\Gamma \to S_{[\varphi]}$ and the right vertical map sends an irreducible representation to the ep twist it lives on.

Fix a hyperspecial vertex $o \in \mathcal{B}^{\mathrm{red}}(G, F)$, and endow G with the corresponding O_F-structure. The construction of (3.1) proceeds in two steps. In the first step, we will explicitly construct a triple $(S_0, [a], [^L j])$, where S_0 is an elliptic maximal torus of G defined over O_F, $[a]$ is an equivalence class of Langlands parameters $a : W_F \to {}^L S_0$, and $[^L j]$ is a $\widehat{G}$-conjugacy class of unramified L-embeddings $^L j : {}^L S_0 \to {}^L G$ such that $[^L j] \circ [a] = [\varphi]$. There will be multiple choices involved in the construction, but we will see in the next section that such a triple is essentially unique, so there will be no need to keep track of these choices. As already mentioned, this step does not require condition 1 of the Langlands parameter, and can be executed for parameters satisfying only conditions 2 and 3. In the second step, we will construct to each $\lambda \in X_*(S_0)$ a quadruple $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$, where $(\psi_\lambda, b_\lambda) : G \to G^\lambda$ is an ep twist and π_λ is an irreducible depth-zero supercuspidal representation of $G^\lambda(F)$. In this step, condition 1 of the Langlands parameter is used. We will show that the quadruples associated to $\lambda, \mu \in X_*(S_0)$ are equivalent if and only λ, μ have the same image in $X_*(S_0)_\Gamma$. The set of equivalence classes of these quadruples will be the L-packet $\Pi_{[\varphi]}$. It is by construction in bijection with $X_*(S_0)_\Gamma$, and we obtain (3.1) by composing this bijection with the group isomorphism $S^D_{[\varphi]} \to X_*(S_0)_\Gamma$ given by $[^L j]$.

Step 1: Let $T \subset G$ be a quasi-split maximal torus such that $o \in \mathcal{A}(T, F)$. Choose a W_F-invariant maximal torus $\widehat{T} \subset \widehat{G}$, which is in duality with T and is part of a Γ-invariant splitting of $\widehat{G}$. Choose a representative $\varphi : W_F \to {}^L G$ of $[\varphi]$ so that $\varphi_0(I_F) \subset \widehat{T}$, where φ_0 is the composition of φ with the projection $^L G \to \widehat{G}$. Then $\varphi_0(W_F) \subset N(\widehat{G}, \widehat{T})$ and we obtain an unramified cocycle

$$
w : W_F \to N(\widehat{G}, \widehat{T}) \twoheadrightarrow \Omega(\widehat{T}, \widehat{G}) \cong \Omega(T, G),
$$

which extends continuously to Γ.

Lemma 3.3.1. *There exists $p_0 \in G(O_{F^u})$ with the properties that the maximal torus $S_0 := \mathrm{Ad}(p_0)T$ is defined over O_F and*

$$
\mathrm{Ad}(p_0) : T^w \to S_0
$$

is an isomorphism of O_F-tori, where T^w is the twist of the O_F-torus T by w.

Proof: We follow the argument of the proof of [BT2, Prop. 5.1.10]. Write G and T for the special fibers of G and T. The reduction map $G(O_{F^u}) \to \mathsf{G}(\overline{k_F})$ provides an isomorphism $\Omega(T, G) \cong \Omega(\mathsf{T}, \mathsf{G})$ and the special fiber of T^w is the same as the twist of T by the image of w under this map. Choose a k_F-embedding $\bar{\xi} : \mathsf{T}^w \to \mathsf{G}$. By [SGA3, exp. 11, Cor 4.2], the functor from O_F-algebras to sets given by $R \mapsto \mathrm{Hom}_{R-\mathrm{grp}}(T^w \times R, G \times R)$ is representable by a

smooth O_F-scheme, hence the map $\mathrm{Hom}_{O_F-\mathrm{grp}}(T^w, G) \to \mathrm{Hom}_{k_F-\mathrm{grp}}(\mathsf{T}^w, \mathsf{G})$ is surjective and we can lift $\bar{\xi}$ to an O_F-homomorphism $\xi : T^w \to G$, which by [SGA3, exp. 9, Cor. 2.5 and 6.6] is a closed embedding. Put $S_0 = \xi(T^w)$, and let $\bar{p}_0 \in \mathsf{G}(\overline{k_F})$ be such that $\mathrm{Ad}(\bar{p}_0) : \mathsf{T}^w \to \mathsf{S}_0$ is an isomorphism of k_F-tori. The scheme $\{g \in G | \mathrm{Ad}(g)T = S_0\}$ is smooth over O_{F^u} and hence we can lift $\bar{p}_0$ to $p_0 \in G(O_{F^u})$. Then the map $\sigma \mapsto p_0^{-1}\sigma(p_0)$ gives an element of $Z^1(\Gamma, \Omega(T, G))$ which equals w, because its image in $Z^1(\Gamma, \Omega(\mathsf{T}, \mathsf{G}))$ equals by construction the image of w. $\qquad\square$

Choose unramified χ-data for $R(S_0, G)$. By the construction of [LS87, 2.6] we obtain a $\widehat{G}$-conjugacy class of L-embeddings $^L S_0 \to {}^L G$. Let $^L j$ be an element of this class with the following properties:

- $^L j(\widehat{S}_0) = \widehat{T}$
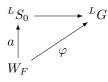
- The cocycles

$$W_F \xrightarrow{\varphi} N(\widehat{T}, \widehat{G}) \rtimes W_F \longrightarrow \Omega(\widehat{T}, \widehat{G})$$

and

$$W_F \xrightarrow{^L j} N(\widehat{T}, \widehat{G}) \rtimes W_F \longrightarrow \Omega(\widehat{T}, \widehat{G})$$

are equal

Then the image of φ is contained in the image of $^L j$ and thus we obtain a factorization

This completes the construction of the triple $(S_0, [a], [^L j])$. The Langlands parameter $[a]$ gives rise to a character $\theta_0 : S_0(F) \to \mathbb{C}^\times$.

Step 2: Let $\lambda \in X_*(S_0)$. We are going to construct the quadruple $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$ as follows: The map $\Phi \mapsto \lambda(\pi)$ extends to a 1-cocycle $W_F \to S_0(\overline{L})$. Its prolongation to $G(\overline{L})$ is basic and will be called b_λ. We claim $b_\lambda \in E(G)$. Let E/F be a finite unramified extension splitting S_0, $\Sigma = \mathrm{Gal}(E/F)$, $n = |\Sigma|$. Recall that S_0 is elliptic, and thus $X_*(S_0)^\Gamma \subset X_*(Z)^\Gamma$, which implies

$$b_\lambda(\Phi^n) = N_\Sigma(\lambda(\pi)) = [N_\Sigma(\lambda)](\pi) \in Z(F),$$

where N_Σ is the norm map for the action of Σ on $X_*(S_0)$. This shows that the image of b_λ in $Z^1(W_F, G_{\mathrm{ad}}(\overline{L}))$ factors through Σ and thus belongs to the image of $Z^1(\Gamma, G_{\mathrm{ad}}(\overline{F}))$, which proves the claim $b_\lambda \in E(G)$. Now b_λ gives rise to the ep twist $(G^\lambda, \psi_\lambda, b_\lambda)$. Put $S_\lambda := \psi_\lambda(S_0)$. Since $\mathrm{Ad}(b_\lambda)$ acts trivially on S_0, we see that

$$\psi_\lambda : S_0 \to S_\lambda$$

is an isomorphism of F-tori. Put $\theta_\lambda = [\psi_\lambda]_* \theta_0$, and let π_λ be the representation $\pi_{G^\lambda, S_\lambda, \theta_\lambda}$ defined in 3.1.

Lemma 3.3.2. *The quadruples* $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$, $(G^\mu, \psi_\mu, b_\mu, \pi_\mu)$ *constructed from two elements* $\lambda, \mu \in X_*(S_0)$ *are equivalent if and only if* λ *and* μ *have the same image in* $X_*(S_0)_\Gamma$.

Proof: By Lemma 3.1.1 the two quadruples are equivalent if and only if there exists a strongly-trivial ep twist $G^\lambda \to G^\mu$ which is equivalent to $(\psi_\mu, b_\mu) \circ (\psi_\lambda, b_\lambda)^{-1}$ and identifies the pairs $(S_\lambda, \theta_\lambda)$ and (S_μ, θ_μ).

Assume first that λ, μ have the same image in $X_*(S_0)_\Gamma$. Using Kottwitz's isomorphism $X_*(S_0)_\Gamma \to \mathbf{B}(S_0)$ we see that there exists $t \in S_0(L)$ such that $b_\lambda = t^{-1} b_\mu \Phi(t)$. Then

$$(\psi_\mu, b_\mu) \circ (\mathrm{Ad}(t), t^{-1}\Phi(t)) \circ (\psi_\lambda, b_\lambda)^{-1}$$

is a strongly-trivial twists $G^\lambda \to G^\mu$ which by construction identifies the pairs $(S_\lambda, \theta_\lambda)$ and (S_μ, θ_μ).

Conversely let $g \in G(L)$ be such that

$$(\psi_\mu, b_\mu) \circ (\mathrm{Ad}(g), g^{-1}\Phi(g)) \circ (\psi_\lambda, b_\lambda)^{-1}$$

is a strongly-trivial twist which identifies (S_λ, b_λ) with (S_μ, b_μ). This implies that $\mathrm{Ad}(g)$ leaves the pair (S_0, θ_0) invariant, which by the regularity of θ_0 is equivalent to $g \in S_0(L)$. Thus b_λ and b_μ have the same image in $\mathbf{B}(S_0)$, and using again the Kottwitz isomorphism we conclude that λ, μ have the same image in $X_*(S_0)_\Gamma$. $\quad\square$

We let $\Pi_{[\varphi]}$ be the set of equivalence classes of quadruples just constructed. Choose any representative ${}^L j$ within the given $\widehat{G}$-conjugacy class, let $\varphi = {}^L j \circ a$ and put $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$. The map ${}^L j$ identifies $[\widehat{S_0}]^\Gamma$ with S_φ, and hence we obtain a bijection
$$S^D_{[\varphi]} \to S^D_\varphi \to X_*(S_0)_\Gamma \to \Pi_{[\varphi]}$$
which is obviously independent of the choice of ${}^L j$.

3.4   Independence of choices and change of base points

This section has a two-fold purpose. On the one hand, we will see that the L-packet $\Pi_{[\varphi]}$ is independent of all choices, while the bijection (3.1) depends only on the $G(F)$-orbit of o. On the other hand, given two choices of hyperspecial vertices $o, o' \in \mathcal{B}^{\mathrm{red}}(G, F)$, we will give the precise relationship between the corresponding bijections.
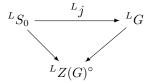
Let $[\varphi]$ be a equivalence class of Langlands parameters satisfying conditions 2 and 3 of Section 3.2. Let $\Xi([\varphi], o)$ be the set of triples $(S_0, [a], [{}^L j])$ where

- $S_0 \subset G$ is an elliptic maximal torus defined over O_F,

- $[a] : W_F \to {}^L S_0$ is an equivalence class of Langlands parameters, and

- $[{}^L j] : {}^L S_0 \to {}^L G$ is a $\widehat{G}$-conjugacy class of unramified L-embeddings,

subject to the conditions

- $[{}^L j] \circ [a] = [\varphi]$, and

- the following diagram commutes

$$
\begin{array}{ccc}
{}^L S_0 & \xrightarrow{\;{}^L j\;} & {}^L G \\
& \searrow \quad \swarrow & \\
& {}^L Z(G)^\circ &
\end{array}
$$

where the diagonal arrows are dual to the natural inclusions.

The triple constructed in Step 1 of the previous section is an element of $\Xi([\varphi], o)$.

If u is an inner automorphism of G defined over F, then we get a map

$$
\Xi([\varphi], o) \to \Xi([\varphi], uo), \quad (S_0, [a], [{}^L j]) \mapsto (u(S_0), [\widehat{u}^{-1} \circ a], [{}^L j \circ \widehat{u}]).
$$

In particular, the group $G(O_F)$ acts on the set $\Xi([\varphi], o)$ by conjugation.

Lemma 3.4.1. *Let $(S_0, [a], [{}^L j])$ and $(S_0', [a'], [{}^L j'])$ be two elements of $\Xi([\varphi], o)$. Then there exists $h \in G(O_F)$ such that $(S_0', [a'], [{}^L j']) = \mathrm{Ad}(h)(S_0, [a], [{}^L j])$.*

Proof: Choose representatives a, a', ${}^L j$ and ${}^L j'$ within the given conjugacy classes such that ${}^L j \circ a = {}^L j' \circ a'$. We claim that the images of ${}^L j$ and ${}^L j'$ coincide. On the one hand, we have ${}^L j(\widehat{S}_0) = \mathrm{Cent}({}^L j \circ a(I_F), \widehat{G})$ and thus we see that ${}^L j(\widehat{S}_0) = {}^L j'(\widehat{S}_0')$. On the other hand, for any $w \in W_F$ we have ${}^L j(w) \in {}^L j(\widehat{S}_0) \cdot {}^L j \circ a(w)$ and the claim follows. Hence we may consider

$$
{}^L \iota : {}^L S_0 \to {}^L S_0'
$$

given by ${}^L j'^{-1} \circ {}^L j$. The restriction of ${}^L \iota$ to $\widehat{S}_0$ is a W_F-equivariant admissible isomorphism $\widehat{S}_0 \to \widehat{S}_0'$, hence there exists $h \in G(\overline{F})$ such that $\mathrm{Ad}(h) : S_0 \to S_0'$ is defined over F and ${}^L \iota|_{\widehat{S}_0} = \widehat{\mathrm{Ad}}(h)^{-1} \rtimes 1$. This h is not yet the element we are looking for. We will successively modify h by multiplying it on the right by elements of $S_0(\overline{F})$ to achieve $h \in G(O_F)$. First, let $h' \in G(O_{F^u})$ be such that $\mathrm{Ad}(h')S_0 = S_0'$ [SGA3, exp 12, 1.7]. Since every element of $\Omega(S_0, G)(\overline{F})$ has a representative in $N(S_0, G)(O_{F^u})$ we see that $h'^{-1}h \in N(S_0, G)(O_{F^u})S_0(\overline{F})$, and conclude that $h \in G(O_{F^u})S_0(\overline{F})$. Modifying h by an element of $S_0(\overline{F})$ we may assume $h \in G(O_{F^u})$. For each $\sigma \in \Gamma$, $h^{-1}\sigma(h)$ belongs to $N(S_0, G)(O_{F^u})$ and acts trivially on S_0, hence belongs to $S_0(O_{F^u})$. Since $H^1(\Gamma, S_0(O_{F^u}))$ is trivial, we can again modify h by an element of $S_0(O_{F^u})$ and achieve $h \in G(O_F)$.

We claim that this h satisfies the statement of the lemma. We already know that $\mathrm{Ad}(h)S_0 = S_0'$, and it will be enough to show that ${}^L j' = {}^L j \circ \widehat{\mathrm{Ad}}(h)$. Put ${}^L j'' = {}^L j' \circ \widehat{\mathrm{Ad}}(h)^{-1}$. By construction, the restriction of ${}^L j^{-1} \circ {}^L j''$ to $\widehat{S}_0$ is the natural inclusion $\widehat{S}_0 \to {}^L S_0$. Let $c : W_F \to {}^L S_0$ be the restriction of ${}^L j^{-1} \circ {}^L j''$ to W_F. This is an unramified Langlands parameter, and we wish to show that the corresponding character χ_c of $S_0(F)$ is trivial. This will imply that ${}^L j''$ and ${}^L j$ are conjugate under ${}^L j(\widehat{S}_0)$, hence $[{}^L j''] = [{}^L j]$ and the proof of the lemma will be complete. Since χ_c is unramified, it is enough by [Kal09, 7.1.1] to show that it is trivial when restricted to $Z(G)^\circ(F) \subset S_0(F)$. If ${}^L p : {}^L S_0 \to {}^L Z(G)^\circ$ is the dual of the natural inclusion $Z(G)^\circ \to S_0$, then the Langlands parameter for $\chi_c|_{Z(G)^\circ(F)}$ is ${}^L p \circ c$. On the other hand, if ${}^L q : {}^L G \to {}^L Z(G)^\circ$ is the dual of the natural inclusion $Z(G)^\circ \to G$, then we obtain, using the third property of our triples, that

$$
{}^L p \circ c = {}^L q \circ {}^L j \circ c = {}^L q \circ {}^L j'' = {}^L p.
$$

$\square$

We are now going to construct for every pair o, o' of hyperspecial vertices in $\mathcal{B}^{\mathrm{red}}(G, F)$ an element $(o, o') \in S_{[\varphi]}$. Choose an element $(S_0, [a], [^L j])$ of $\Xi([\varphi], o)$. We obtain a group homomorphism

$$G_{\mathrm{ad}}(F) \to H^1(F, Z(G)) \to H^1(F, S_0) \to \mathbf{B}(S_0) \to X_*(S_0)_\Gamma \to S^D_{[\varphi]},$$

where the first map is the connecting homomorphism, the second is induced by the natural inclusion, the third is the injection from [Kot85, 1.8], the fourth is the isomorphism constructed in [Kot85, §2], and the fifth is the map induced by $^L j$. We let $(o, o') \in S^D_{[\varphi]}$ be the image of any $g_{\mathrm{ad}} \in G_{\mathrm{ad}}(F)$ such that $g_{\mathrm{ad}} o = o'$ under this homomorphism. It is clear that if $o' \in G(F)o$, then $(o, o') = 1$.

Lemma 3.4.2. *This map is well defined, and if $o, o', o'' \in \mathcal{B}^{\mathrm{red}}(G, F)$ are three hyperspecial vertices, then*
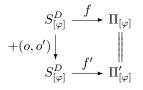
$$(o, o'') = (o, o') \cdot (o', o'').$$

Proof: The independence of the choice of $(S_0, [a], [^L j])$ follows from Lemma 3.4.1. To show that the choice of g_{ad} is also irrelevant, let $g'_{\mathrm{ad}} \in G_{\mathrm{ad}}(F)$ be another element with $g'_{\mathrm{ad}} o = o'$, and consider $h_{\mathrm{ad}} := g^{-1}_{\mathrm{ad}} g'_{\mathrm{ad}}$. Endow G and G_{ad} with the O_F-structure corresponding to o. Then $S'_0 = \mathrm{Ad}(h_{\mathrm{ad}}) S_0$ is another maximal torus of G defined over O_F. Let $h \in G(\overline{F})$ be a lift of h_{ad}. Arguing as in the proof of Lemma 3.4.1 we can write $h = h_1 s_1$ where $h_1 \in G(O_F)$ and $s_1 \in S_0(\overline{F})$. Then the image of h_{ad} in $H^1(F, S_0)$ is represented by the cocycle $\sigma \mapsto s^{-1}_1 h^{-1}_1 \sigma(h_1) \sigma(s_1)$. The cocycle $\sigma \mapsto h^{-1}_1 \sigma(h_1)$ takes values in $S_0(O_{F^u})$, and is thus cohomologically trivial. We conclude that the elements g_{ad} and g'_{ad} map to the same element of $S^D_{[\varphi]}$.

The multiplicative property follows easily from the fact that if $(S_0, [a], [^L j]) \in \Xi([\varphi], o)$ and $g_{\mathrm{ad}} \in G_{\mathrm{ad}}(F)$ is such that $g_{\mathrm{ad}} o = o'$, then $\mathrm{Ad}(g_{\mathrm{ad}})(S_0, [a], [^L j]) \in \Xi([\varphi], o')$ and the maps $H^1(F, Z(G)) \to S^D_{[\varphi]}$ induced by these two triples are the same. $\square$

We now assume that $[\varphi]$ satisfies the conditions 1-3 of Section 3.2.

Proposition 3.4.3. *Let $o, o' \in \mathcal{B}^{\mathrm{red}}(G, F)$ be hyperspecial, and choose $(S_0, [a], [^L j]) \in \Xi([\varphi], o)$ and $(S'_0, [a'], [^L j']) \in \Xi([\varphi], o')$. If $f : S^D_{[\varphi]} \to \Pi_{[\varphi]}$ and $f' : S^D_{[\varphi]} \to \Pi'_{[\varphi]}$ are the corresponding versions of (3.1), then $\Pi_{[\varphi]} = \Pi'_{[\varphi]}$ and we have a commutative diagram*

$$
\begin{array}{ccc}
S^D_{[\varphi]} & \xrightarrow{\ f\ } & \Pi_{[\varphi]} \\
{\scriptstyle +(o, o')}\Big\downarrow & & \Big\| \\
S^D_{[\varphi]} & \xrightarrow{\ f'\ } & \Pi'_{[\varphi]}
\end{array}
$$

In particular, the bijection f depends only on the $G(F)$-orbit of o.

Proof: Fix representatives $a, {}^L j$ in the classes $[a], [^L j]$ and put $\varphi = {}^L j \circ a$. As in step 2, $^L j$ gives a map $S^D_\varphi \to X_*(S_0)_\Gamma$ which we will denote by $^L j^D$. Let $\rho \in S^D_\varphi$ and $\lambda \in X_*(S_0)$ be such that the image of ρ under $^L j^D$ coincides with the class of λ in $X_*(S_0)_\Gamma$.

Let $g \in G(\overline{F})$ be such that its image in G_{ad} is defined over F and maps o to o'. By Lemma 3.4.1 we may choose g so that $(S'_0, [a'], [^L j']) = \mathrm{Ad}(g)(S_0, [a], [^L j])$.

Let $\lambda_g \in X_*(S_0)$ be an element whose image in $X_*(S_0)_\Gamma$ corresponds to (o, o') under ${}^L j^D$. Then $\lambda' := \mathrm{Ad}(g)[\lambda + \lambda_g]$ is an element of $X_*(S_0')$, ${}^L j' := {}^L j \circ \widehat{\mathrm{Ad}}(g)^{-1}$ is a representative of the class $[{}^L j']$, and the image of $\rho + (o, o')$ under ${}^L j'^D$ coincides with the class of λ' in $X_*(S_0')_\Gamma$.

As in step 2 of the previous section, we obtain from (S_0, θ_0, λ) a quadruple $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$. This quadruple is the image of ρ under f. In the same way, we obtain from $(S_0', \theta_0', \lambda')$ a quadruple $(G^{\lambda'}, \psi_{\lambda'}, b_{\lambda'}, \pi_{\lambda'})$, which is the image of $\rho + (o, o')$ under f'. We claim that the two quadruples are equivalent. Indeed, there exists an $s \in S_0(\overline{F})$ with $gs \in G(F^u)$ and $(gs)^{-1}\Phi(gs) = \lambda_g(\pi)^{-1}$. A straightforward computation shows that

$$(\psi_{\lambda'}, b_{\lambda'}) \circ (\mathrm{Ad}(g), (gs)^{-1}\Phi(gs)) \circ (\psi_\lambda, b_\lambda)^{-1}$$

is a strongly-trivial twist which identifies $(S_\lambda, \theta_\lambda)$ with $(S_{\lambda'}, \theta_{\lambda'})$ and hence π_λ with $\pi_{\lambda'}$. $\square$

3.5 Relation to the construction of DeBacker and Reeder

We continue with G, ${}^L G$, and $[\varphi]$ as in 3.3. In [DR09] the authors choose a certain representative φ of $[\varphi]$ and construct an L-packet Π_φ' on all pure inner forms of G as well as a bijection

$$\pi_0(S_\varphi)^D \to \Pi_\varphi'.$$

The goal of this section is to prove the following

Theorem 3.5.1. *The set Π_φ' can be naturally identified with a subset of $\Pi_{[\varphi]}$, and we have a commutative diagram*

$$
\begin{array}{ccc}
\pi_0(S_\varphi)^D & \longrightarrow & \Pi_\varphi' \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
S_{[\varphi]}^D & \longrightarrow & \Pi_{[\varphi]}
\end{array}
$$

Let us very briefly recall their construction.

First, the authors choose a quasi-split maximal torus $T \subset G$, a hyperspecial vertex $o \in \mathcal{A}^{\mathrm{red}}(T, F)$, and a Γ-invariant maximal torus $\widehat{T} \subset \widehat{G}$ dual to T. They write ${}^L G = \widehat{G} \rtimes W_F$ and choose φ within its equivalence class so that $\varphi_0(I_F) \subset \widehat{T}$, where φ_0 is the projection of φ to $\widehat{G}$. Then

$$w : W_F \xrightarrow{\varphi_0} N(\widehat{T}, \widehat{G}) \longrightarrow \Omega(\widehat{T}, \widehat{G}) \xrightarrow{\cong} \Omega(T, G)$$

is an unramified cocycle. They choose an element $\dot{w}$ of finite order inside $N(T, G)(O_{F^u})$ which lifts $w(\Phi)$.

Let T^w be the twist of T by w. As an L-group for T^w we can take

$$ {}^L T^w = \widehat{T} \rtimes_\varphi W_F,$$

where W_F acts on $\widehat{T}$ via its image under φ. In [DR09, 4.3], a Langlands parameter $\varphi_T : W_F \to {}^L T^w$ is constructed, by letting $\varphi_T|_{I_F} = \varphi|_{I_F}$, and letting $\varphi_T(\Phi) = \tau \rtimes \Phi$ where $\tau \in \widehat{T}$ is any element whose image under

$$\widehat{T} \to \widehat{G} \to [\widehat{G}]_{\mathrm{ab}}$$

belongs to the Φ-twisted conjugacy class of $\varphi(\Phi)$. The Langlands parameter φ_T corresponds to a character $\theta : T^w(F) \to \mathbb{C}^\times$.

Next, for every $\mu \in X_*(T^w)$ whose image in $X_*(T^w)_\Gamma$ is torsion, DeBacker and Reeder use in [DR09, 2,4.4] the combinatorics of the Bruhat-Tits building of G to construct an unramified cocycle $u_\mu \in Z^1(\Gamma, G)$. Let $(\psi'_\mu, u_\mu) : G \to G^{u_\mu}$ be the corresponding pure inner twist. Furthermore, they construct $p_\mu \in G(F^u)$ such that both the maximal torus $T_\mu := \psi'_\mu \mathrm{Ad}(p_\mu)T$ and the isomorphism $\psi'_\mu \mathrm{Ad}(p_\mu) : T^w \to T_\mu$ are defined over F. Via this isomorphism, the character θ can be transported to a character $\theta_\mu : T_\mu(F) \to \mathbb{C}^\times$, and one obtains a representation $\pi'_\mu := \pi_{G^{u_\mu}, T_\mu, \chi_\mu}$ as in Section 3.1. It is argued in [DR09, 4.5.3] and [Ree08, 6.1] that the equivalence class of the quadruple $(G^{u_\mu}, \psi'_\mu, u_\mu, \pi'_\mu)$ depends only on the image of μ in $X_*(T^w)_\Gamma$. The set $\Pi'_{[\varphi]}$ is then the set of equivalence classes of such quadruples, and we obtain the bijection

$$\pi_0(S_\varphi)^D = \pi_0([\widehat{T}^w]^\Gamma)^D = X_*(T^w)_\Gamma \to \Pi'_{[\varphi]}.$$

We will not review the details of the construction. The crucial property that will be important for us is [DR09, 2.7.(8)], namely

$$p_\mu^{-1} u_\mu \Phi(p_\mu) = \mu(\pi)\dot{w}.$$

Moreover, $u_0 = 1$, and $p_0 \in G(O_{F^u})$.

Lemma 3.5.2. *There exists an L-embedding $^L j_0 : {}^L T_0 \to {}^L G$ extending the admissible isomorphism $\widehat{\mathrm{Ad}}(p_0) : \widehat{T}_0 \to \widehat{T}$, such that the triple $(T_0, [a_0], [{}^L j_0])$ with $a_0 = \widehat{\mathrm{Ad}}(p_0)^{-1} \circ \varphi_T$ belongs to $\Xi([\varphi], o)$.*

Proof: The extension of $\widehat{\mathrm{Ad}}(p_0)$ to an L-embedding $^L j_0$ is provided by the construction [LS87, 2.6] after choosing unramified χ-data. The torus T_0 is conjugate to T by p_0, hence defined over O_{F^u}. We only need to check that $[{}^L j_0 \circ a_0] = [\varphi]$. It is clear that this equality holds when both sides are restricted to I_F. To show that both sides are also equal when evaluated at Φ, we argue as follows. Let $a_1 : W_F \to {}^L T_0$ be a homomorphism such that $[{}^L j_0 \circ a_1] = [\varphi]$ (see the argument in Step 1 of 3.3). The Langlands parameters a_0 and a_1 coincide on inertia, hence the difference of characters on $T_0(F)$ that they correspond to is unramified. To show that this difference is in fact trivial, and hence $[a_0] = [a_1]$, we argue again as in the proof of Lemma 3.4.1. By [Kal09, 7.1.1] it is enough to show that the difference is trivial when restricted to $Z(G)^\circ(F) \subset S_0(F)$. This restriction of the character corresponding to a_0 has the Langlands parameter

$$^L p \circ [a_0] = {}^L q \circ [{}^L j_0 \circ a_0] = {}^L q \circ \varphi$$

by construction of φ_T. The restriction of the character corresponding to a_1 has the Langlands parameter

$$^L p \circ [a_1] = {}^L q \circ [{}^L j_0 \circ a_1] = {}^L q \circ \varphi$$

by construction of a_1. □

Proof: (of Theorem 3.5.1)
We keep the choices of $o, T, \widehat{T}, \varphi$ made so far. Let $\rho \in \pi_0(S_\varphi)^D$, and let $\mu \in X_*(T^w)$ be an element whose image in $X_*(T^w)_\Gamma$ equals the image of ρ under the identification $\pi_0(S_\varphi)^D = [X_*(T^w)_\Gamma]_{\mathrm{tor}}$. Let $\lambda = \mathrm{Ad}(p_0)\mu \in X_*(T_0)$. By the

preceding Lemma, we can use the triple $(T_0, [a_0], [{}^L j_0])$ for our construction of (3.1). Then we see that the image of ρ under the map

$$\pi_0(S_\varphi)^D \to S_\varphi^D \to X_*(T_0)_\Gamma$$

given by ${}^L j_0$ equals the class of λ. Thus we need to show that the quadruple $(G^{u_\mu}, \psi_\mu', u_\mu, \pi_\mu')$ constructed by DeBacker-Reeder resp. Reeder and the quadruple $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$ constructed in Section 3.3 are equivalent. By Lemma 3.1.1 we need to find a strongly trivial twist

$$(G^\lambda, \psi_\lambda, b_\lambda) \to (G^{u_\mu}, \psi_\mu', u_\mu),$$

which carries the pair $(S_\lambda, \theta_\lambda)$ to the pair (T_μ, θ_μ). We claim that

$$(\psi_\mu', u_\mu) \circ \mathrm{Ad}(p_\mu p_0^{-1}) \circ (\psi_\lambda, b_\lambda)^{-1}$$

is a strongly-trivial twist. Computing the cocycle of this twist we obtain

$$
\begin{aligned}
&\ \psi_\lambda(\mathrm{Ad}[p_\mu p_0^{-1}]^{-1}(u_\mu)(p_\mu p_0^{-1})^{-1}\Phi(p_\mu p_0^{-1})b_\lambda^{-1}) \\
=&\ \psi_\lambda(p_0 p_\mu^{-1} u_\mu \Phi(p_\mu)\Phi(p_0^{-1})b_\lambda^{-1}) \\
=&\ \psi_\lambda(p_0 \mu(\pi)\dot w \Phi(p_0^{-1})b_\lambda) \\
=&\ \psi_\lambda(\lambda(\pi)\mathrm{Ad}[p_0](\dot w \Phi(p_0^{-1})p_0)b_\lambda^{-1}) \\
=&\ 1.
\end{aligned}
$$

If $\theta_0 : T_0(F) \to \mathbb{C}^\times$ is the character corresponding to the parameter a_0, then we have $\theta_0 = \mathrm{Ad}(p_0)\theta$, and hence the above twists carries the pair $(S_\lambda, \theta_\lambda)$ to the pair (T_μ, θ_μ). $\qquad\square$

3.6 Generic representations

We saw in Proposition 3.4.3 that the bijection (3.1) depends only the the $G(F)$-orbit of the chosen hyperspecial vertex o. The image of the trivial character under this bijection is a generic representation, and the set of generic characters with respect to which this representation is generic can also be read off from the vertex o.

For a hyperspecial vertex o' and $r \geq 0$ consider the set of pairs

$$\mathrm{Gen}[o', r] = \{(B, \psi)\},$$

where B is a Borel subgroup of G such that the reduced apartment of some maximal torus of B contains o', and $\psi : B_u(F) \to \mathbb{C}^\times$ is a character of generic depth r at o'. Let

$$\mathrm{Gen}[G(F)o, r] = \bigcup_{o' \in G(F)o} \mathrm{Gen}[o', r].$$

Fact 3.6.1. *Let o a hyperspecial vertex, and π the image of the trivial character on $S_{[\varphi]}$ under the version of* (3.1) *corresponding to o. Then for each $(B, \psi) \in \mathrm{Gen}[G(F)o, 0]$ we have $\mathrm{Hom}_{B_u(F)}(\pi, \psi) \neq 0$.*

Proof: This is the content of [DR09, §6]. $\qquad\square$

Recall that an F-splitting of G is a triple $(T, B, \{X_\alpha\})$ where (T, B) is a Borel pair, and $X_\alpha \in \mathfrak{g}_\alpha(\overline{F})$ is a non-trivial element for each simple root of T in B, and

the set $\{X_\alpha\}$ is Γ-invariant. Any pair (T, B) can be extended to an F-splitting. Moreover, the isomorphism $\mathbb{G}_a \to \mathfrak{g}_\alpha$ given by $1 \mapsto X_\alpha$ exponentiates to a homomorphism $x_\alpha : \mathbb{G}_a \to G$ whose image is the unique 1-dimensional subgroup of G which T normalizes and acts on by the character α. If G is defined over O_F, an F-splitting of G will be called an O_F-splitting if T is defined over O_F, and each homomorphism x_α is defined over O_F and induces a non-trivial homomorphism on the special fibers.

Let
$$\mathrm{Spl}[o'] = \{(T, B, \{X_\alpha\})\}$$
consist of the O_F-splittings of G for the O_F-structure defined by o', and let
$$\mathrm{Spl}[G(F)o] = \bigcup_{o' \in G(F)o} \mathrm{Spl}[o'].$$

Fix a character $\Lambda : F \to \mathbb{C}^\times$ which is trivial on $\pi^{r+1} O_F$ but non-trivial on $\pi^r O_F$. An element $(T, B, \{X_\alpha\}) \in \mathrm{Spl}[o']$ gives rise to an element $(B, \psi) \in \mathrm{Gen}[o', r]$ as follows. The map
$$\prod_{\alpha \in \Delta(T,B)} \mathbb{G}_a \to B_u \to B_u^{\mathrm{ab}}, \qquad (\xi_\alpha) \mapsto \prod x_\alpha(\xi_\alpha)$$

is an isomorphism of $\overline{F}$-groups. Composing its inverse with the summation map gives a homomorphism of F-groups $B_u^{\mathrm{ab}} \to \mathbb{G}_a$, which then composed with Λ provides a generic depth r character $\psi : B_u(F) \to \mathbb{C}^\times$.

Proposition 3.6.2. *The map $(T, B, \{X_\alpha\}) \mapsto (B, \psi)$ is $G_{\mathrm{ad}}(F)$-equivariant. It induces a surjection*
$$\mathrm{Spl}[G(F)o] \to \mathrm{Gen}[G(F)o, r]$$
whose fiber over a given (B, ψ) is a $B_u(F)$-orbit. In particular, it induces a bijection
$$\mathrm{Spl}[G(F)o]/\mathrm{Ad}(G(F)) \to \mathrm{Gen}[G(F)o, r]/\mathrm{Ad}(G(F)).$$

Proof: The $G_{\mathrm{ad}}(F)$-equivariance is obvious from the construction. To show surjectivity, fix $(B, \psi) \in \mathrm{Gen}[o, r]$. Choose any F-splitting $(T', B', \{X'_\alpha\})$ and let (B', ψ') be the corresponding generic character. Since (B, ψ) and (B', ψ') are conjugate under $G_{\mathrm{ad}}(F)$, we see that there exists a splitting $(T, B, \{X_\alpha\})$ mapping to (B, ψ). Conjugating this splitting by $B_u(F)$ we may assume that the reduced apartment of T contains o. The fact that the depths of ψ and Λ are equal forces $(T, B, \{X_\alpha\}) \in \mathrm{Spl}[o]$.

Now consider two splittings $(T, B, \{X_\alpha\})$ and $(T', B, \{X'_\alpha\})$ in the fiber over (B, ψ). Conjugating by $B_u(F)$ we may assume that $T = T'$. There exists a unique element $t \in T_{\mathrm{ad}}(F)$ sending $\{X_\alpha\}$ to $\{X'_\alpha\}$. We wish to show that in fact $t = 1$. By assumption, $\mathrm{Ad}(t)$ preserves ψ. Let $\alpha \in \Delta(T, B)$ and let E/F be the field of definition of α. If $U_{(\alpha)}$ is the relative root subgroup of G defined by the restriction (α) of α to the split part of T, then
$$E \to [U_{(\alpha)}]_{\mathrm{ab}}(F), \quad \xi \mapsto \prod_{\sigma \in \mathrm{Gal}(E/F)} x_{\sigma\alpha}(\sigma\xi)$$

is an isomorphism of groups. Composing ψ with this isomorphism yields a non-trivial character on E. The automorphism $\mathrm{Ad}(t)$ pre-composes this character with multiplication by $\alpha(t) \in E^\times$. Thus, the assumption that $\mathrm{Ad}(t)$ preserves ψ implies that $\alpha(t) = 1$ for all $\alpha \in \Delta(T, B)$, hence $t = 1$. $\qquad\square$

We keep the notation established so far. In particular, G is an unramified connected reductive group defined over F, ^LG is an L-group for G, and $[\varphi]$ is an equivalence class of Langlands parameters satisfying conditions 1-3 of Section 3.2.

4.1 Preparatory lemmas

Let J be a connected reductive group defined over F with Lie algebra j, B a conjugation-invariant bilinear form on j(F), and $\chi : F \to \mathbb{C}^\times$ an additive character. To this data, one can define the Weil constant $\gamma_\chi(B)$, see [Wal95, VIII]. If J' is an inner form, or an endoscopic group of J, then Waldspurger shows in loc. cit. how to transfer B to a conjutation-invariant bilinear form B' on j$'(F)$, and the ratio $\gamma_\chi(B)\gamma_\chi(B')^{-1}$ plays an important role in Waldspurger's work on endoscopy for p-adic Lie algebras. In order to prove the stability and endoscopic transfer of our depth-zero packets, we need to know that when J and J' split over F^u, this ratio can be expressed in terms of the F-split ranks of J and J'. This has been worked out when J and J' are pure inner forms in [DR09, §12], and when J is quasi-split and J' is an endoscopic group in [Kal09, §4]. Here we build on the arguments of [DR09, §12] to handle the case when J and J' are general inner forms. Of course it is enough to assume that J is quasi-split.

Lemma 4.1.1. *Let J be an unramified connected reductive group defined over F and $\psi : J \to J'$ an inner twist. Let B be an $\mathrm{Ad}(J(F))$-invariant bilinear form on $\mathrm{Lie}(J)(F)$, and $\chi : F \to \mathbb{C}^\times$ an additive character. Then*

$$e(J') = \gamma_\chi(B)\gamma_\chi(B')^{-1},$$

where B' is the corresponding bilinear form on $\mathrm{Lie}(J')(F)$, and e is the Kottwitz-sign.

Proof: Let $T' \subset J'$ be an unramified elliptic maximal torus, which exists by [Deb06, 2.4]. Up to equivalence of ψ we may assume that the maximal torus $T := \psi^{-1}(T')$ is defined over F and $\psi : T \to T'$ is an F-isomorphism. Recall that an orbit of Γ in $R(T, J)$ is called symmetric if it is invariant under multiplication by -1. Let $\mathrm{Sym}(T)$ be a set of representatives for the symmetric orbits. For each $\alpha \in \mathrm{Sym}(T)$, let F_α be the fixed field of the stabilizer of α in Γ, and $F_{\pm\alpha}$ be the fixed field of the stabilizer of $\{\alpha, -\alpha\}$ in Γ. The root subspace $\mathfrak{j}_\alpha \subset \mathfrak{j}$ is defined over F_α and we may choose $E_\alpha \in \mathfrak{j}_\alpha(F_\alpha)$. It was shown by Waldspurger [Wal95, VIII.5] that

$$\gamma_\chi(B) = \gamma_\chi(B|_{\mathfrak{t}(F)}) \prod_{\alpha \in \mathrm{Sym}(T)} \gamma_{\chi_\alpha}(q_\alpha),$$

where $\chi_\alpha : F_{\pm\alpha} \to \mathbb{C}^\times$ is the character $\chi \circ \mathrm{tr}_{F_{\pm\alpha}/F}$, and $q_\alpha : F_\alpha \to F_{\pm\alpha}$ is the quadratic form $\lambda \mapsto \mathrm{Nm}_{F_\alpha/F_{\pm\alpha}}(\lambda)B(E_\alpha, E_{-\alpha})$. Let d be the level of χ (the largest integer such that χ is trivial on the ideal $\pi^{-d}O_F$). Since $\mathrm{tr}_{F_{\pm\alpha}/F} : O_{F_{\pm\alpha}} \to O_F$ is surjective, d is also the level of χ_α. It is known ([JL70, Proof of Lemma 1.2]) that

$$\gamma_{\chi_\alpha}(q_\alpha) = (-1)^{d+\mathrm{val}(B(E_\alpha, E_{-\alpha}))}.$$

A corresponding formula holds for $\gamma_\psi(B')$ as well and since ψ provides an isomorphism $(\mathfrak{t}(F), B) \to (\mathfrak{t}'(F), B')$ and a bijection $\mathrm{Sym}(T) \to \mathrm{Sym}(T')$ we obtain

$$\gamma_\chi(B)\gamma_\chi(B')^{-1} = \prod_{\alpha \in \mathrm{Sym}(T)} (-1)^{\mathrm{val}\{B(E_\alpha, E_{-\alpha})\}B'(E'_{\psi(\alpha)}, E'_{-\psi(\alpha)})\}}.$$

There exists $c_\alpha \in F_\alpha^\times$ such that $E'_{\psi(\alpha)} = c_\alpha\psi(E_\alpha)$ and the right hand side becomes $\prod(-1)^{\mathrm{val}(c_\alpha c_{-\alpha})}$.

Let $a_\sigma \in Z^1(\Gamma, J_{\mathrm{ad}})$ be such that $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(a_\sigma)$. By our assumption $a_\sigma \in Z^1(\Gamma, T_{\mathrm{ad}})$. There exists $\lambda \in X_*(T_{\mathrm{ad}})$ with $a_\Phi = \lambda(\pi)$ [Kal09, 2.3.3]. The computations of [DR09, 12.3.2+12.3.3] show that

$$\prod_{\alpha \in \mathrm{Sym}(T)} (-1)^{\mathrm{val}(c_\alpha c_{-\alpha})} = (-1)^{\langle \lambda, 2\rho \rangle}.$$

where 2ρ is the sum of any choice of positive roots in $R(T, J)$.

By definition, $e(J')$ is the image of a_σ under

$$H^1(F, T_{\mathrm{ad}}) \to H^1(F, J_{\mathrm{ad}}) \xrightarrow{\partial} H^2(F, Z(J_{\mathrm{sc}})) \xrightarrow{\rho} H^2(F, \mathbb{G}_m) \xrightarrow{\exp(2\pi i \cdot \mathrm{inv})} \mathbb{C}^\times.$$

Tate-Nakayama-duality identifies the profinite completion of $H^0(F, X^*(\mathbb{G}_m))$ with the character group of $H^2(F, \mathbb{G}_m)$. One has

$$\exp(2\pi i \cdot \mathrm{inv}(x)) = 1(x), \quad \text{for } x \in H^2(F, \mathbb{G}_m), 1 \in \mathbb{Z} \xrightarrow{\cong} H^0(F, X^*(\mathbb{G}_m)).$$

Hence, applying Tate-Nakayama-duality to the above sequence of arrows we see that $e(J')$ is equal to the pairing of a with the image of 1 under the dual sequence

$$H^0(F, X^*(\mathbb{G}_m)) \to H^0(F, X^*(Z(J_{\mathrm{sc}}))) \to H^1(F, X^*(T_{\mathrm{ad}}))$$

This image is represented by the cocycle $\Phi \mapsto \Phi(\rho) - \rho$. Since a is the image of $-\lambda$ under the Tate-Nakayama isomorphism $H^{-1}_{\mathrm{Tate}}(F, X_*(T_{\mathrm{ad}})) \to H^1(F, T_{\mathrm{ad}})$ one sees by applying [Kal09, 2.3.2] that the pairing of a and $\Phi \mapsto \Phi(\rho) - \rho$ is $\exp(2\pi i \langle \lambda, \rho \rangle)$. $\qquad \square$

4.2 The unstable character

Let $\varphi : W_F \to {}^L G$ be a Langlands parameter as in Section 3.2 whose depth is zero. We fix a hyperspecial vertex $o \in \mathcal{B}^{\mathrm{red}}(G, F)$ and obtain the corresponding version of the bijection (3.1). To each $\rho \in S^D_{[\varphi]}$, this bijection assigns an equivalence class of quadruples $\{(G_b, \psi, b, \pi)\}$. Let $\rho, \rho' \in S^D_{[\varphi]}$ and (G_b, ψ, b, π) and $(G_{b'}, \psi', b', \pi')$ be representatives within the corresponding equivalence classes. We will call an ep twist $G_b \to G_{b'}$ *admissible* if it is equivalent to $\psi' \circ \psi^{-1}$. If ρ, ρ' have the same image $\bar{\rho}$ in $[Z(\widehat{G})^\Gamma]^D$, then there exists an admissible strongly trivial ep twists $G_b \to G_{b'}$, and every such induces the same bijection $\mathrm{Cl}(G_{b,\mathrm{sr}}(F)) \to \mathrm{Cl}(G_{b',\mathrm{sr}}(F))$ between the conjugacy classes of strongly regular semi-simple elements. Hence we may speak of the set $\mathrm{Cl}_{\bar{\rho}}$, and the characters of π and π' provide functions Θ_ρ and $\Theta_{\rho'}$ on $\mathrm{Cl}_{\bar{\rho}}$. These functions are equal if and only if $\rho = \rho'$. Moreover, the Kottwitz signs of G_b and $G_{b'}$ are the same and we can write $e_{\bar{\rho}}$ for them.

Definition 4.2.1. *Let* $t \in S_{[\varphi]}$. *We construct a function* $\Theta^t_{[\varphi]}$ *on* $\bigsqcup_{\xi \in [Z(\widehat{G})^\Gamma]^D} \mathrm{Cl}_\xi$,

which is given on each Cl_ξ *by*

$$\Theta^t_{[\varphi]} = e_\xi \sum_\rho \rho(t)\Theta_\rho,$$

where ρ *runs over the fiber of* $S^D_{[\varphi]} \to [Z(\widehat{G})^\Gamma]^D$ *over* ξ.

Fix an element $(S_0, [a], [^L j]) \in \Xi([\varphi], o)$. We obtain a map

$$X_*(S_0)_\Gamma \xrightarrow{\cong} S^D_{[\varphi]}. \tag{4.1}$$

For $\lambda \in X_*(S_0)$, let $\rho_\lambda \in S^D_{[\varphi]}$ be its image. Then the quadruple $(G^\lambda, \psi_\lambda, b_\lambda, \pi_\lambda)$ constructed in Section 3.3 is a representative of the element of $\Pi_{[\varphi]}$ corresponding to ρ_λ under (3.1).

Proposition 4.2.2. *Let* $Q_0 \in \mathrm{Lie}(S_0)(F)$ *be a regular semi-simple element. For any* $\gamma \in G^\lambda_{\mathrm{sr}}(F)_0$ *and* $z \in Z(F)$, *the value of* $\Theta^t_{[\varphi]}(z\gamma)$ *is given by*

$$\epsilon(G, A_G)\theta_0(z) \sum_P [\varphi_{Q_0, P}]_* \theta_0(\gamma_s) \sum_Q \langle \mathrm{inv}(Q_0, Q), t \rangle^{-1} R(G^\lambda_{\gamma_s}, S_Q, 1)(\gamma_u)$$

where P *runs over a set of representatives for the* $G^\lambda_{\gamma_s}$-*stable classes of elements of* $\mathrm{Lie}(G^\lambda_{\gamma_s})(F)$ *which are* G^λ-*stably conjugate to* Q_0, *and* Q *runs over a set of representatives for the* $G^\lambda_{\gamma_s}(F)$-*conjugacy classes inside the* $G^\lambda_{\gamma_s}$-*stable class of* P, *and* $\langle \mathrm{inv}(Q_0, Q), - \rangle$ *is the image of* $\mathrm{inv}(Q_0, Q)$ *under the map* $\boldsymbol{B}(S_0) \to S^D_{[\varphi]}$.

Proof: Let $M \subset X_*(S_0)$ be a set of representatives for the fiber of

$$X_*(S_0)_\Gamma \xrightarrow{(4.1)} S^D_{[\varphi]} \longrightarrow [Z(\widehat{G})^\Gamma]^D$$

through λ. For each $\mu \in M$, and let $\psi_{\mu, \lambda} : G^\mu \to G^\lambda$ be a strongly trivial admissible ep twist. Put $(S'_\mu, \theta'_\mu) = \psi_{\mu, \lambda}(S_\mu, \theta_\mu)$. Then Θ_{ρ_μ} equals the character of $\pi_{G^\lambda, S'_\mu, \theta'_\mu}$. Hence using Lemma 3.1.2 and noting that θ_0 and θ'_μ coincide on $Z(F)$ we obtain

$$\Theta^t_{[\varphi]}(z\gamma) = \epsilon(G, A_G)\theta_0(z) \sum_{\mu \in M} \rho_\mu(t) \sum_Q R(G^\lambda_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_\mu, Q}]_* \theta'_\mu(\gamma_s),$$

where $Q_\mu = \psi_{\mu, \lambda}\psi_\mu(Q_0)$ and Q runs over any set of representatives for the $G^\lambda_{\gamma_s}(F)$-conjugacy classes inside the $G^\lambda(F)$-conjugacy class of Q_μ. We have

$$\rho_\mu(t) = \langle b_\mu^{-1}, t \rangle = \langle \mathrm{inv}(Q_0, Q_\mu), t \rangle^{-1} = \langle \mathrm{inv}(Q_0, Q), t \rangle^{-1}$$

The reason for the $-$-sign is that b_μ is the image of $-\mu$ under the isomorphism $X_*(S_0)_\Gamma \to \boldsymbol{B}(S_0)$, and the last equality holds because Q_μ and Q are $G^\lambda(F)$-conjugate. Recalling $\theta'_\mu = [\psi_{\lambda, \mu}\psi_\mu]_* \theta_0$ we arrive at

$$\Theta^t_{[\varphi]}(z\gamma) = \epsilon(G, A_G)\theta_0(z) \sum_\mu \sum_Q \langle \mathrm{inv}(Q_0, Q), t \rangle^{-1} R(G^\lambda_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_0, Q}]_* \theta_0(\gamma_s)$$

Applying the Lie algebra version of [Kal09, 2.1.5] in our context we see that as μ runs over the fiber of $X_*(S_0)_\Gamma \to \boldsymbol{B}(G)_b$ through λ, Q_μ runs over a set

of representatives for the $G^\lambda(F)$-rational classes inside the intersection of the stable class of Q_0 with $\mathrm{Lie}(G^\lambda)(F)$. Thus in the double sum, Q runs over a set of representatives for the $G^\lambda_{\gamma_s}(F)$-classes inside the intersection of the stable class of Q_0 with $\mathrm{Lie}(G^\lambda)(F)$. We can rewrite this sum as

$$\Theta^t_{[\varphi]}(z\gamma) = \epsilon(G, A_G)\theta_0(z)\sum_P\sum_Q \langle \mathrm{inv}(Q_0, Q), t\rangle^{-1} R(G^\lambda_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_0, Q}]_*\theta_0(\gamma_s)$$

where P and Q run as in the statement of the lemma. To conclude the proof, we only need to notice that if Q, Q' are $G^\lambda_{\gamma_s}$-conjugate, then $[\varphi_{Q_0, Q}]_*\theta_0(\gamma_s) = [\varphi_{Q_0, Q'}]_*\theta_0(\gamma_s)$. $\qquad\square$

Lemma 4.2.3. *Let $\gamma^\lambda \in G^\lambda(F)_0$ be strongly regular semi-simple and assume that $\gamma^\lambda_s \in S(F)$ for some maximal torus $S \subset G^\lambda$ which is stably conjugate to S_0. Then there exists $\gamma \in G(F)$ stably conjugate to γ^λ with $\gamma_s \in S_0(F)$.*

Proof: The argument is the same as for [Kal09, 7.2.2], but instead of using $\mathrm{Ad}(q_0 q_\lambda^{-1})$ we use any admissible ep twist sending S to S_0. $\qquad\square$

4.3 Stability and endoscopy

In this section we need to impose further restrictions on F. Let n_G be the smallest dimension of a faithful representation of G, e the ramification degree of $F/\mathbb{Q}_p$, e_G the maximum ramification degree over $\mathbb{Q}_p$ of a splitting field of a maximal torus of G, and $\nu(G)$ be the number of positive roots of G. For Theorem 4.3.1, we require $q_F \geq \nu(G)$ and $p \geq (2+e)n_G$. For Theorem 4.3.3 we require in addition $p \geq 2 + e_G$ and $p \geq (2+e)n_H$ (H the corresponding endoscopic group).

Theorem 4.3.1. *The function $\Theta^1_{[\varphi]}$ is stable. That is, if $\gamma^\lambda \in G^\lambda_{\mathrm{sr}}(F), \gamma^\mu \in G^\mu_{\mathrm{sr}}(F)$ are stably conjugate, where $\lambda, \mu \in X_*(S_0)$, then $\Theta^1_{[\varphi]}(\gamma^\lambda) = \Theta^1_{[\varphi]}(\gamma^\mu)$.*

Proof: We have $\gamma^\lambda \in Z(F)G^\lambda_{\mathrm{sr}}(F)_0$ if and only if $\gamma^\mu \in Z(F)G^\mu_{\mathrm{sr}}(F)_0$. We may assume this is the case, for otherwise both sides of the equation vanish [DR09, 9.6.3]. Write $\gamma^\lambda = zsu$ with $z \in Z(F)$, s top. semi-simple and u top. unipotent. By Lemma 4.2.3 there exists $\gamma \in G(F)$ stably conjugate to γ^λ, with corresponding decomposition $\gamma = ztv$ and $t \in S_0(F)$. It will be enough to show the proposition in the case $\mu = 0$ and $\gamma^\mu = \gamma$.

Choose a semi-simple element $Q_0 \in \mathrm{Lie}(S_0)(O_F)$ with strongly-regular reduction; such an element exists, see [DR09, 12.4.2]. Let $(\psi, b) : G^\lambda \to G$ be an admissible ep twist sending γ^λ to γ. The ep twist (ψ, b) restricts to an ep twist

$$(\psi, b) : G^\lambda_s \to G_t$$

and provides an injection of stable classes, both on the level of the groups and their Lie algebras. We will denote this injection again by ψ. Since elliptic tori transfer to inner forms, this injection restricts to a bijection between those stable classes which are stably conjugate to Q_0 under G^λ resp. G. From Proposition 4.2.2 and [DR09, 12.4.3] we have

$$\Theta^1_{[\varphi]}(\gamma^\lambda) = \epsilon(G, A_G)\epsilon(G^\lambda_s, A_{G^\lambda_s})\theta_0(z)\sum_{P_\lambda}[\varphi_{Q_0, P_\lambda}]_*\theta_0(s)\widehat{S}^{G^\lambda_s}_{P_\lambda}(\log(u))$$

$$\Theta^1_{[\varphi]}(\gamma) = \epsilon(G, A_G)\epsilon(G_t, A_{G_t})\theta_0(z) \sum_P [\varphi_{Q_0, P}]_* \theta_0(t) \widehat{S}^{G_t}_P(\log(v))$$

where P_λ runs over a set of representatives for the G^λ_s-stable classes of elements of $\mathfrak{g}^\lambda_s(F)$ which are stably conjugate to Q_0, P runs over a set of representatives for the G_t-stable classes of elements of $\mathfrak{g}_t(F)$ which are stably conjugate to Q_0, and $\widehat{S}^{G^\lambda_s}_{P_\lambda}$ denotes the Fourier transform of the stable orbital integral of P_λ in $\mathfrak{g}^\lambda_s$. From [Wal97, Conj 1.2] and Lemma 4.1.1 we know

$$\widehat{S}^{G^\lambda_s}_{P_\lambda}(\log(u)) = \epsilon(G_t, G^\lambda_s)\widehat{S}^{G_t}_{\psi(P_\lambda)}(\log(v))$$

Since $A_{G^\lambda_s} = A_{G_t}$, the lemma follows. $\qquad\square$

Let $(H, s, {}^L\eta)$ be an extended unramified endoscopic triple for G. Let $[\varphi^H] : W_F \to {}^L H$ be a Langlands parameter and put $[\varphi] = [{}^L\eta] \circ [\varphi]$. Assume that φ is a parameter of the type described in Section 3.2. Then so is φ^H.

Fix a hyperspecial vertex $o \in \mathcal{B}^{\mathrm{red}}(G, F)$. To each element of $\mathrm{Gen}[G(F)o, 0]$, we have the Whittaker normalization of the transfer factor for (G, H), as explained in [KS99, 5.3]. In fact, the particular choice of element is irrelevant:

Proposition 4.3.2. *Each element of $\mathrm{Gen}[G(F)o, 0]$ determines the same Whittaker normalization of the transfer factor.*

Proof: Let (B, ψ) and (B', ψ') belong to $\mathrm{Gen}[G(F)o, 0]$. Conjugating the Whittaker data by $G(F)$ has no effect on the transfer factors, so we may assume $B = B'$, as well as $(B, \psi), (B, \psi') \in \mathrm{Gen}[o, 0]$. Let $(T, B, \{X_\alpha\})$ and $(T, B, \{X'_\alpha\})$ be two elements of $\mathrm{Spl}[o, 0]$ which give rise to ψ resp. ψ' as in Proposition 3.6.2. The normalization of transfer factor corresponding to the Whittaker data (B, ψ) differs from the normalization corresponding to the splitting $(T, B, \{X_\alpha\})$ by a term depending on Λ. Hence it is enough to show that the two splittings $(T, B, \{X_\alpha\})$ and $(T, B, \{X'_\alpha\})$ lead to the same normalization, which is the content of [Hal93, 7.2]. $\qquad\square$

Thus the Whittaker normalization of the transfer factor depends only on the $G(F)$-orbit of o. We will write Δ_o for it. Of course, it also depends on Λ and the extended triple $(H, s, {}^L\eta)$, but those we assume fixed.

Let $\lambda \in X_*(S_0)$ and $(G^\lambda, \psi_\lambda, b_\lambda)$ be the corresponding ep twist. Let $\Delta_{o,\lambda}$ be the normalization of the transfer factor for (G^λ, H) constructed from Δ_o in Section 2.2. With respect to this transfer factor we have the endoscopic lift of the stable character of the L-packet $\Pi^H_{[\varphi^H]}$

$$\mathrm{Lift}^{G^\lambda, o}_H \Theta^1_{[\varphi^H],0}(\gamma^\lambda) := \sum_{\gamma^H} \Delta_{o,\lambda}(\gamma^H, \gamma^\lambda) \frac{D^H(\gamma^H)^2}{D^{G^\lambda}(\gamma^\lambda)^2} \Theta^1_{[\varphi^H],0}(\gamma^H),$$

where γ^H runs over the stable conjugacy classes of strongly regular semi-simple elements of $H(F)$ and D are the usual Weyl discriminants.

Theorem 4.3.3. *For any $\gamma^\lambda \in G_{\mathrm{sr}}(F)$ we have*

$$\mathrm{Lift}^{G^\lambda, o}_H \Theta^1_{[\varphi^H],0}(\gamma^\lambda) = \Theta^s_{[\varphi]}(\gamma^\lambda).$$

Proof: The argument in [Kal09, §7] works just as well in this case. The only modifications necessary are that references to Section 2, Proposition 6.2.2, and Lemma 7.2.2 in loc. cit, and to [DR09, §12.3], have to be replaced with references to Section 2, Proposition 4.2.2, Lemma 4.2.3 and Lemma 4.1.1 in this paper. $\qquad\square$

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