The Ellipse

$$x(t) = aa\cos(t),\ y(t) = bb\sin(t),\ 0 \le t \le 2\pi$$

3DXM-SUGGESTION:
Select in the Action Menu: *Show Osculating Circles with Normals*. In the Animate Menu try the default *Morph*.
For related curves see: Parabola, Hyperbola, Conic Sections and their ATOs.

The Ellipse is shown together with the so called *Leitkreis construction* of the curve and its tangent, see below. This construction assumes that the constants aa and bb are positive. The larger of the two is called the semi-major axis length, the smaller one is the semi-minor axis length.
The Ellipse is also the set of points satisfying the following implicit equation: $(x/aa)^2 + (y/bb)^2 = 1$.

A geometric definition of the Ellipse, that can be used to shape flower beds is:

> An Ellipse is the set of points for which the **sum of the distances** from two focal

> points is a constant L equal to twice the semi-major axis length.

A gardener connects the two focal points by a cord of length L, pulls the cord tight with a stick which then draws the boundary of the flower bed with the stick. Another version of this definition is:

> An Ellipse is the set of points which have **equal distance** from a circle of radius L and a (focal) point inside the circle.

Both these definitions are illustrated in the program.

The normal to an ellipse at any point bisects the angle made by the two lines joining that point to the foci. This says that rays coming out of one focal point are reflected off the ellipse towards the other focal point. Therefore one can build elliptically shaped "whispering galleries", where a word spoken softly at one focal point can be heard only close to the other focal point.

To add a simple proof we show that the tangent leaves the ellipse on one side; more precisely, we show that for every other point on the tangent the sum of the distances to the two focal points F_1, F_2 is more than the length L of the major axis. (In the display: $F = F_2$.) Pick any point Q on the tangent, join it to the two focal points and reflect the segment QF in the tangent, giving another segment QS. Now F_1QS is a radial straight segment only if Q is the point of tangency—otherwise F_1QS is by the triangle inequality longer than the radius F_1S (of length L) of the circle around F_1.

The evolute of an ellipse, i.e., the curve enveloped by the normals of the ellipse—see Action Menu: *Draw osculating circles with normals*, is a generalized Astroid, it is less symmetric than the true Astroid.

An Ellipse can also be obtained by a rolling construction: Inside a circle of radius aa another circle of radius $r := hh = 0.5aa$ rolls and traces the Ellipse with a stick of radius $R := ii \cdot r$, see Plane Curves Menu: *Circle* and select from the Action Menu: *Show Generalized Cycloids*. The parametric equation resulting from this construction is:

$x(t) = (R + r)\cos(t)$
$y(t) = (R - r)\sin(t)$

This is related to the visualization of the complex map $z \to z + 1/z$ in Polar Coordinates, the image of the circle of radius R is such an ellipse with $r = 1/R$.

Such rolling constructions are reached with the Plane Curves Menu entry: *Circle* and then the Action Menu *Draw Generalized Cycloids* or with *Epi- and Hypocycloids*. Recall that negative values of the rolling radius hh gives curves on the outside, positive radii ($hh < aa$) on the inside of the fixed circle.

Other rolling curves are:

Cycloid, Astroid, Deltoid, Cardioid, Limacon, Nephroid, Epi- and Hypocycloids.

H.K.