Prof. Dr. Daniel Huybrechts
Dr. René Mboro

Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 9

Exercise 43. (Noether normalization over rings, 3 points)
Assume an injective ring homomorphism $A \hookrightarrow B$ makes an integral domain B a finite type A-algebra. Show that then there exist $0 \neq a \in A$ and elements $y_1, \ldots, y_n \in B$ that are algebraically independent over A such that the localizaion B_a is integral over $A[y_1, \ldots, y_n]_a$.

Exercise 44. (Finite type $\mathbb{Z}$-algebras are Jacobson, 3 points)
Assume A is a Jacobson ring and B is an A-algebra. Show that B is a Jacobson ring as well if B is of finite type over A or integral over A.

Exercise 45. (Finite fields, 3 points)
Show that any field that is finitely generated as a $\mathbb{Z}$-algebra is in fact a finite field.
Hint: One might need to show that there is no surjective $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{Q}$.

Exercise 46. (Family of polynomials without common zeros, 3 points)
Let $f_1, \ldots, f_k \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials without any common zero $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Show that there exist $g_1, \ldots, g_k \in \mathbb{Z}[x_1, \ldots, x_n]$ with $0 \neq g_1 f_1 + \ldots + g_k f_k \in \mathbb{Z}$. Is this still true if $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$?

Exercise 47. (Noether normalization via linear projections, 4 points)
Let $V(\mathfrak{a}) \subset \mathbb{A}^3_k$ with $\mathfrak{a} = (y - z^2, xz - y^2) \subset k[x, y, z]$. Determine explicitly a linear projection $V(\mathfrak{a}) \to \mathbb{A}^1_k$ which is finite, closed, and surjective.

Exercise 48. (Valuation rings, 3 points)
Consider a field extension $K \subset L$, B a valuation ring with quotient field L, and let $A := K \cap B$. Prove the following statements:

(i) A is a valuation ring with quotient field K.

(ii) If L/K is algebraic and B is not a field, then also A is not a field.

Am Donnerstag, den 11.06.2020 findet eine Versammlung aller Mathematikstudierenden (Fachschaftsvollversammlung) statt. Alle weiteren Informationen findet ihr unter `www.fsmath.uni-bonn.de`

On Thursday, 11.06.2020 there will be an online-meeting of all math students (Fachschaftsvollversammlung). All further information can be found at `www.fsmath.uni-bonn.de`

Solutions to be handed in before Tuesday June 15, 4pm.