Solutions for exercises, Algebra I (Commutative Algebra) – Week 11

Exercise 55. (Union of associated prime ideals, 3 points)
Passing to the quotient $A/\mathfrak{a}$, the setting becomes (0) admits a primary decomposition $\cap_{i=1}^n \mathfrak{q}_i$ and we want to show that $\cup_i \sqrt{\mathfrak{q}_i} = \{a \in A,\ a \text{ is a zero} - \text{divisor}\}$. Let $a \neq 0$ be a zero-divisor and $b \neq 0$ such that $ab = 0 \in \cap_{i=1}^n \sqrt{qq_i}$. If $\forall i,\ a \notin \sqrt{\mathfrak{q}_i}$, those ideals being primes, we have $b \in \sqrt{\mathfrak{q}_i},\ \forall i$. If there is a i_0 such that $b \notin \mathfrak{q}_{i_0}$ then $0 \neq \overline{b} \in A/\mathfrak{q}_{i_0}$ and $\overline{a}\overline{b} = 0 \in A/\mathfrak{q}_{i_0}$ i.e. $\overline{a}$ is a zero-divisor in $A/\mathfrak{q}_{i_0}$, which, since $\mathfrak{q}_{i_0}$ is primary means that $\overline{a}$ is nilpotent i.e. $a^k \in \mathfrak{q}_{i_0}$ for some $k > 0$; contradicting $a \notin \sqrt{\mathfrak{q}_{i_0}}$.
Otherwise, $b \in \mathfrak{q}_i$, $\forall i$ i.e. $b \in \cap_{i=1}^n \mathfrak{q}_i = (0)$, $b = 0$; contradiction. So $a \in \sqrt{\mathfrak{q}_i}$ for some i.

Conversely, according to Proposition 14.8, for a given i, there is a $a \in A$ such that $\sqrt{\mathfrak{q}_i} = \sqrt{(0:a)} = \sqrt{\mathrm{Ann}(a)}$ (in particular $a \neq 0$). So $\forall x \in \sqrt{\mathfrak{q}_i}$, there is a $k > 0$ such that $x^k \in \mathrm{Ann}(a)$ i.e. $x^k a = 0$ and $a \neq 0$. So there is a $k - 1 \geq \ell \geq 0$ such that $x^\ell a \neq 0$ but $x(x^\ell a) = x^{\ell+1} a = 0$. So x is a zero-divisor.

Exercise 56. (Products of coprime ideals, 2 points)
The case $n = 2$ is proved in the lecture notes (see p.6 footnote 3). So let $n \geq 2$ be a integer such that for any set $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ of n pairwise coprime ideals (i.e. $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ for any $i \neq j$) we have the equality: $\prod_{i=1}^n \mathfrak{a}_i = \cap_{i=1}^n \mathfrak{a}_i$.
Let $\mathfrak{a}_1, \dots, \mathfrak{a}_{n+1}$ be a set of $n+1$ pairwise coprime ideals. Then by induction hypothesis $\prod_{i=1}^n \mathfrak{a}_i = \cap_{i=1}^n \mathfrak{a}_i$. Since $\mathfrak{a}_i + \mathfrak{a}_{n+1} = (1)$ for any $i \leq n$, we can write $1 = a_i + x_i$ where $a_i \in \mathfrak{a}_i$ and $x_i \in \mathfrak{a}_{n+1}$. Taking the product, we get

$$1 = \Pi_{i=1}^n a_i + \sum_{i=0}^{n-1} \sum_{K \subset \{1,\dots,n\}, \#K = i} (\prod_{i \in K} a_i)(\prod_{j \in \{1,\dots,n\} \setminus K} x_j).$$

Now $\Pi_{i=1}^n a_i \in \mathfrak{a}_1 \cdots \mathfrak{a}_n$ and in the second term, $\{1, \dots, n\} \setminus K$ is always nonempty; thus $(\prod_{i \in K} a_i)(\prod_{j \in \{1,\dots,n\} \setminus K} x_j) \in \mathfrak{a}_{n+1}$. So $(1) = \Pi_{i=1}^n \mathfrak{a}_i + \mathfrak{a}_{n+1}$ i.e. $\Pi_{i=1}^n \mathfrak{a}_i$ and $\mathfrak{a}_{n+1}$ are coprime so by the case $n = 2$, $\Pi_{i=1}^n \mathfrak{a}_i \cap \mathfrak{a}_{n+1} = \Pi_{i=1}^{n+1} \mathfrak{a}_i$ but we also had (induction hypothesis) $\Pi_{i=1}^n \mathfrak{a}_i = \cap_{i=1}^n \mathfrak{a}_i$ so $\Pi_{i=1}^{n+1} \mathfrak{a}_i = \Pi_{i=1}^n \mathfrak{a}_i \cap \mathfrak{a}_{n+1} = \cap_{i=1}^{n+1} \mathfrak{a}_i$; completing the induction step.

Exercise 57. (Primary decomposition, 4 points)

1. Using Lecture 10, we have

$$\begin{aligned} V(\mathfrak{a}) = V(xy) \cap V(x - yz) &= (V(x) \cup V(y)) \cap V(x - yz) \\ &= (V(x) \cap V(x - yz)) \cup (V(y) \cap V(x - yz)) \\ &= V((x) + (x - yz)) \cup V((y) + (x - yz)) \\ &= V((x) + (yz)) \cup V((y) + (x)) \\ &= (V(x) \cap V(yz)) \cup V(x, y) \\ &= (V(x) \cap (V(y) \cup V(z))) \cup V(x, y) \\ &= (V(x, y) \cup V(x, z)) \cup V(x, y) \\ &= V(x, y) \cup V(x, z) \end{aligned}$$

Solutions to be handed in before Monday June 29, 4pm.

2. From $V(\mathfrak{a}) = V(x,y) \cup V(x,z) = V((x,y) \cap (x,z))$ we get $\sqrt{\mathfrak{a}} = \sqrt{(x,y) \cap (x,z)} = \sqrt{(x,y)} \cap \sqrt{(x,z)} = (x,y) \cap (x,z)$ since (x,y) and (x,z) are prime ideals (the associated quotients are resp. $k[z]$ and $k[y]$ which are both integral domains).

3. We have $(xy, x-yz) = (xy-y(x-yz), x-yz) = (zy^2, x-yz)$. We can look at $k[x,y,z]/\mathfrak{a}$ as two successive quotients $A' := k[x,y,z]/(x-yz)$ and $A'/(zy^2) \simeq k[x,y,z]/\mathfrak{a}$. Now, $A' \simeq k[yz,y,z] \simeq k[y,z]$ and in $A'/(zy^2)$, $(0) = (y^2) \cap (z)$; thus $\mathfrak{a} = (y^2) \cap (z) \bmod (x-yz)$ i.e. $\mathfrak{a} = (x-yz, y^2) \cap (x-yz, z)$. But $(x-yz, z) = (x,z)$ so it is a prime (hence primary) ideal.
Again looking at successive quotients we get $k[x,y,z]/(y^2, x-yz) \simeq k[y,z]/(y^2)$. Any element of $k[y,z]/(y^2)$ can be written uniquely as $f_1(z) + yf_2(z)$ with $f_i \in k[z]$; so such element is a zero-divisor there is $g_1 + yg_2 \neq 0$ such that $f_1g_1 + y(f_1g_2 + f_2g_1) = 0 \bmod (y^2)$. Thus we must have $f_1g_1 = 0 \in k[z]$ and $f_1g_2 + f_2g_1 = 0 \in k[z]$; which with the condition that g_1 or g_2 is not 0 (and $k[z]$ is integral) yields $f_1 = 0$. So $f = yf_2 \bmod (y^2)$; but then f is nilpotent since $f^2 = y^2 f_2^2 \bmod (y^2) = 0 \bmod (y^2)$. So $(y^2, x-yz)$ is primary. Since $z \notin (y^2, x-yz)$ and $y^2 \notin (x,z)$ the decomposition is minimal.

Exercise 58. (Example of a primary ideal, 3 points)
By definition of $\mathfrak{m}$, for any polynomial $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ we have $f = a_0 \bmod 2$ thus $f \notin \mathfrak{m}$ if and only if $f(0) = a_0$ is odd. So given $f = \sum_{i=0}^{n} a_i x^i \notin \mathfrak{m}$, we can write a_0 as $2k+1$. But then $1 = f - x(\sum_{i=1}^{n} a_i x^{i-1}) - 2k$ i.e. $1 \in \mathfrak{m} + (f)$. Thus $\mathfrak{m}$ is maximal.
We have $\mathbb{Z}[x]/(4,x) \simeq \mathbb{Z}[x]/(4) \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x) \simeq \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x) \simeq \mathbb{Z}/4\mathbb{Z}$ (using tensor identity 5 of sheet 6 for the first isomorphism) and the isomorphism is given by $f \mapsto f(0) \bmod 4$. So only zero-divisors in $\mathbb{Z}[x]/\mathfrak{q} \simeq \mathbb{Z}/4\mathbb{Z}$ are $\overline{0}$ and $\overline{2}$ which are nilpotent i.e. $\mathfrak{q}$ is a primary ideal. Moreover $\sqrt{\mathfrak{q}}$ is the contraction of the nilradical of $\mathbb{Z}[x]/\mathfrak{q}$; since $\mathfrak{R}_{\mathbb{Z}[x]/\mathfrak{q}} \simeq (\overline{2})$ we get $\sqrt{\mathfrak{q}} = (2,x) = \mathfrak{m}$.
We have $\mathfrak{m}^2 = (4, 2x, x^2)$ and $\mathfrak{m}^k = (2^k, 2^{k-1}x, \ldots, 2^{k-i}x^i, \ldots, x^k)$ for $k \geq 2$ which are readily seen not to contain $x \in \mathfrak{q}$.

Exercise 59. (Case of radical ideals, 2 points)
Let $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition; we get $\mathfrak{a} = \sqrt{\mathfrak{a}} = \cap_{i=1}^{n} \sqrt{\mathfrak{q}_i}$ and the $\sqrt{\mathfrak{q}_i}$ are prime ideals. If there is a non minimal prime ideals among the $\sqrt{\mathfrak{q}_i}$'s, we can assume $\sqrt{\mathfrak{q}_1} \subset \sqrt{\mathfrak{q}_2}$. Then $\mathfrak{a} = \cap_{i=1}^{n} \sqrt{\mathfrak{q}_i} = \cap_{i\neq 2} \sqrt{\mathfrak{q}_i}$ and for any $a \in \cap_{i\neq 2} \mathfrak{q}_i \subset \cap_{i\neq 2} \sqrt{\mathfrak{q}_i} = \mathfrak{a}$ since $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$, we get $a \in \mathfrak{q}_i$ for any i; in particular $a \in \mathfrak{q}_2$ thus $\cap_{i\neq 2} \mathfrak{q}_i \subset \mathfrak{q}_2$ i.e. the primary decomposition is not minimal; contradiction. So all the $\sqrt{\mathfrak{q}_i}$ are minimal.

Exercise 60. (Primary decomposition, 3 points)
We have $A/\mathfrak{p}_1 \simeq k[z]$, $A/\mathfrak{p}_2 \simeq k[y]$ and $A/\mathfrak{m} \simeq k$; so $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals (in particular primary) and $\mathfrak{m}$ is maximal. So by Lemma 14.4 (ii), $\mathfrak{m}^2$ is primary.
We have $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2 = (x^2, xz, xy, yz)$; thus we immediately get $\mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. Moreover $\mathfrak{m}^2 = (x^2, xy, xz, y^2, yz, z^2)$ thus from the generators we see that $\mathfrak{a} \subset \mathfrak{m}^2$ i.e. $\mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Let $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $f \in \mathfrak{m}^2$ we can write $f = x^2 f_1 + xyf_2 + xzf_3 + y^2 f_4 + yzf_5 + z^2 f_6$; then $f \in \mathfrak{p}_1$ (since $x^2 = xx, xy, xz, y^2, yz, yz \in \mathfrak{p}_1$) if and only if $f_6 \in \mathfrak{p}_1$; write it $f_6 = xg_1 + yg_2$. Likewise $f \in \mathfrak{p}_2$ if and only if $f_4 \in \mathfrak{p}_2$; write it as $f_4 = xg_3 + zg_4$. then

$$f = x^2 f_1 + xy(f_2 + yg_3) + xz(f_3 + zf_6) + yz(f_5 + yf_4 + zf_6) \in \mathfrak{a}.$$

So $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$.
Likewise $\mathfrak{p}_1 \cap \mathfrak{m}^2 = (x^2, xy, xz, yz, y^2)$ which is not contained in $\mathfrak{p}_2$ because $y^2 \notin \mathfrak{p}_2$.
Likewise $\mathfrak{p}_2 \cap \mathfrak{m}^2 = (x^2, xy, xz, yz, z^2)$ which is not contained in $\mathfrak{p}_1$ because $z^2 \notin \mathfrak{p}_1$.
An element $f \in \mathfrak{p}_2$ can be written $f = xf_1 + zf_2$ and it is in $\mathfrak{p}_1$ if and only if $f_2 \in \mathfrak{p}_1$ so $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, xz, yz)$. In particular, we see $x \notin \mathfrak{m}^2$ so $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq\subset \mathfrak{m}^2$. Thus $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition.
We have $\mathfrak{p}_i \subsetneq \sqrt{\mathfrak{m}^2} = \mathfrak{m}$, so $\mathfrak{m}$ is an embedded component and $\mathfrak{p}_i$ are isolated components.