Exercise Session 13

① Let X be an AV over some field k, $g = \dim X$.

(a) Let $\varphi, \psi \in \mathrm{End}(X)$, $L \in \mathrm{Pic}(X)$. Then there are $L_0, L_1, L_2 \in \mathrm{Pic}(X)$ s.t.

$\forall n \in \mathbb{Z}$:

$$(\psi + n\varphi)^* L = L_0 \otimes L_1^{n} \otimes L_2^{n(n-1)/2}$$

Thm of the Cube for $f = \psi + n\varphi$, $g = h = \varphi$:

$$\underbrace{(\psi + (n+2)\varphi)^* L}_{f+g+h} = \underbrace{(\psi + (n+1)\varphi)^* L}_{f+g} \otimes \underbrace{(2\varphi)^* L}_{g+h} \otimes \underbrace{(\psi + (n+1)\varphi)^* L}_{f+h} \otimes$$

$$\otimes \underbrace{(\psi + n\varphi)^* L^{-1}}_{f} \otimes \underbrace{\varphi^* L^{-1}}_{g} \otimes \underbrace{\varphi^* L^{-1}}_{h}$$

Let $L_{(n)} := (\psi + n\varphi)^* L$.

$$\rightsquigarrow L_{(n+2)} = L_{(n+1)}^{2} \otimes L_{(n)}^{-1} \otimes \underbrace{\left((2\varphi)^* L \otimes \varphi^* L^{-2}\right)}_{=: M}$$

Let

$n=0$ $\quad L_0 = \psi^* L$

$n=1$ $\quad L_1 = (\varphi + \psi)^* L \otimes \psi^* L^{-1}$

$n=2$ $\quad L_2 = (\psi + 2\varphi)^* L \otimes (\varphi + \psi)^* L^{-2} \otimes \psi^* L^{2} \otimes \varphi^* L^{-1}$

Need to check recurrence relation holds for explicit formula:

$$\cancel{L_0} \otimes \cancel{L_1^{n+2}} \otimes L_2^{(n+2)(n+1)/2} = \cancel{L_0^{2}} \otimes \cancel{L_1^{2n+2}} \otimes L_2^{n(n+1)} \otimes \cancel{L_0^{-1}} \otimes \cancel{L_1^{-n}} \otimes L_2^{-n(n-1)/2} \otimes M$$

$\Leftrightarrow L_2 = M$

Know the formula is true for $n=2 \Rightarrow L_2 = M \Rightarrow$ formula holds $\forall n$.

(b) $\deg: \mathrm{End}(X) \to \mathbb{Z}$ extends to a polynomial function $\mathrm{End}^{\mathbb{Q}}(X) \to \mathbb{Q}$.

($\deg \varphi = 0$ if φ is not surjective)

Claim: $\forall \psi, \varphi \in \mathrm{End}(X)$, $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto \deg(\psi + n\varphi)$ is polynomial

Proof: Fix an ample line bundle L on X.

$\rightsquigarrow \forall \varphi \in \mathrm{End}(X)$: $(\varphi^* L)^g = \deg \varphi \cdot (L)^g$

φ not surj: both sides $=0$

φ surj $\Rightarrow \varphi$ fin. loc. free (miracle flatness)

$\rightsquigarrow$ formula holds by end of lecture 22

L ample $\Rightarrow (L)^g \neq 0 \Rightarrow \deg \varphi = \dfrac{(\varphi^* L)^g}{(L)^g}$

$$\rightsquigarrow \deg(\psi + n\varphi) = ((\psi + n\varphi)^* L)^g / (L)^g$$

$$= (L_0 \otimes L_1^n \otimes L_2^{n(n-1)/2})^g / (L)^g$$

is polynomial by multilinearity of $(-)^g$. $\square$

$\rightsquigarrow$ Remains to see: For $\psi, \varphi \in \mathrm{End}^{\mathbb{Q}}(X)$,

$$\mathbb{Q} \to \mathbb{Q}, \quad n \mapsto \deg(\psi + n\varphi)$$

is polynomial

- Note $\deg(n\varphi) = n^{2g} \deg\varphi$

  $\rightsquigarrow$ w.l.o.g. $\varphi, \psi \in \mathrm{End}(X)$

- Write $P_\psi : n \mapsto \deg(\psi + n\varphi)$, $n \in \mathbb{Q}$. Then $\forall \frac{a}{b} \in \mathbb{Q}$

$$P_\psi\left(\frac{a}{b}\right) = b^{-2g} P_{b\psi}(a)$$

is polynomial in a (for fixed b).

$\Rightarrow$ necessarily all these polynomials are equal, independent of b.

$\square$

(c) Let $\ell \neq \mathrm{char}\, k$ be a prime. Let

$$T_\ell X = \varprojlim_n X[\ell^n](\bar{k}) \cong \mathbb{Z}_\ell^{2g}$$

The map

$$\mathbb{Z}_\ell \otimes \mathrm{End}(X) \hookrightarrow \mathrm{End}(T_\ell X)$$

is injective.

<u>Claim</u>: Let Y be another AV, $M \subseteq \mathrm{Hom}(X, Y)$ be fin. gen. subgrp.

Then $\mathbb{Q}M \cap \mathrm{Hom}(X,Y)$ is still fin. gen.

<u>Proof</u>: We say that an AV Z is simple if it contains no AV other than 0 and itself.

By Poincaré Reducibility (page 17 on lecture 25) we can write

$$X \sim \prod_i X_i, \quad Y \sim \prod_j Y_j$$

for simple AV's X_i, Y_j.

$$\Rightarrow \mathrm{Hom}^0(X,Y) = \prod_{i,j} \mathrm{Hom}^0(X_i, Y_j)$$ ← Hom^0 only depends on isogeny class.

Reason: If $f: X \to Y$ is isogeny then $\exists g: Y \to X$ isogeny s.t. $gf = [\deg f]$.

(Get $X/_{\ker f} \xrightarrow{\sim} Y$.

$$\begin{tikzcd} X \arrow[d, "\deg f"'] \arrow[r] & X/_{\ker f} = Y \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

as $\ker f$ is killed by $[\deg f]$.)

$\rightsquigarrow$ w.l.o.g. X, Y are simple.

1. If $X \not\sim Y$ then $\mathrm{Hom}(X,Y) = 0$.

2. If $X \sim Y$ then $\mathrm{Hom}^0(X,Y) = \mathrm{End}^0(X,X)$

$\Rightarrow$ w.l.o.g. $X = Y$.

In this case, every $0 \neq \varphi \in \mathrm{End}(X)$ is an isogeny $\Rightarrow \deg \varphi \neq 0$.

By (6), $\deg: \mathrm{End}^0(X) \to \mathbb{Q}$ is continuous.

$\Rightarrow U = \{\varphi \in \mathbb{Q}\cdot M \mid |\deg \varphi| < 1\} \subseteq \mathbb{Q}\cdot M\}$

is open nbhd of 0 in $\mathbb{Q}\cdot M$, and $U \cap \mathrm{End}(X) = 0$.

$\Rightarrow \mathrm{End}(X) \cap \mathbb{Q}\cdot M \subseteq \mathbb{Q}\cdot M$ is discrete

$\rightsquigarrow$ fin. gen. $\square$

Rest of proof: Same as for elliptic curves.

<u>Rmk on Exercise 12.1(6)</u>: Proof of $\mathcal{O}(Z_1)\cdot\mathcal{O}(Z_2) = \mathrm{len}(Z_1 \cap Z_2)$.

Had reason why it should be true. Problem: Could not find a "good" meromorphic section of $\mathcal{O}(Z_i)$. But

$$I(Z_i) \hookrightarrow \mathcal{O}_X \quad \rightsquigarrow \quad \mathcal{O}_X^\vee = \mathcal{O}_X \xrightarrow{s_i} \mathcal{O}(Z_i) \;\hat{=}\; s_i \in \mathcal{O}(Z_i)$$

Take these s_i and everything works :)