

# Towards a $d$ -minimal Pila-Wilkie theorem

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# Introduction

Model theory, after a modern conception proposed by Hrushovski [14], is the

"geography of tame mathematics".

The informal expression 'tame' is most often defined negatively by the absence of 'wild' phenomena in mathematical structures. What is regarded as 'wild', depends on the context; this includes Gödelian phenomena like incompleteness and undecidability as well as counterintuitive behavior from a geometrical point of view such as the existence of space-filling curves. The task of model theory, following the above slogan, is to identify and analyze the structures that are well-behaved, counting on the fact that the results will also find applications in nontame settings.

In this thesis we explore geometrical consequences of *d-minimality*, a tameness condition on expansions of the real field, in the context of generalizing the *Pila-Wilkie theorem*. In the landscape of mathematics the following is located at the interface of logic and geometry, demonstrating the ability of model theory to bridge the gap between logic and 'classical' mathematics by analyzing familiar structures from a logical perspective, focusing on the interplay of syntax and semantics.

The real ordered field is one of the most basic structures a lot of mathematics is built on. By the fundamental Tarski-Seidenberg theorem, its theory admits quantifier elimination - a typical tameness result. The theorem can be reformulated in a geometric fashion, stating that any coordinate projection of a semialgebraic set is semialgebraic. Semialgebraic sets are finite unions of sets in  $\mathbb{R}^n$  defined by polynomial equations and inequalities. The theorem implies that the structure of the real field is tame in many senses. Not only it proves decidability of its theory, it has also deep geometrical consequences for the definable sets (the semialgebraic sets). This includes a cell decomposition theorem, stratification theorems and a well-behaved notion of dimension excluding space-filling semialgebraic curves.

Thinking of model theory as the geography of tame mathematics, the questions arise: What is it, in essence, that ensures the tameness of this structure? What can be said about the tameness of other structures expanding the real line?

It is well known that an expansion of the real field that defines the integers is 'wild' - its theory is undecidable and the definable sets become arbitrarily complicated, comprising the whole projective hierarchy. In contrast, in the 1980s, driven by the work of Lou van den Dries, a model-theoretic condition was found that implies that a structure shares the tame properties of the real field - *o-minimality*.

A structure  $\mathcal{R}$  expanding the real line is *o-minimal*, if any definable subset of  $\mathbb{R}$  is a finite union of points and intervals. This condition on the structure of definable subsets of the real line has striking consequences for the geometry of definable sets in any dimension.

The rich theory for o-minimal structures soon led to the opinion, that o-minimal geometry is a realization of the "tame topology" Grothendieck had called for in 1984 [23]. By now, many examples of o-minimal structures have been found, including the expansion of the real field by the exponential function or by 'restricted analytic functions'. A boost was given to the recognition of o-minimality by the *Pila-Wilkie theorem* in 2006 [33]. This theorem deals with counting rational points in definable sets in o-minimal structures.

**Theorem** (Pila-Wilkie Theorem). *Let  $X$  be a definable set in some o-minimal expansion of the real field. Then for all  $\varepsilon \in \mathbb{R}_+$  there is a constant  $c \in \mathbb{R}_+$  s.t. for all  $T \in \mathbb{R}$ ,  $T \geq 1$ , there are not more than  $c \cdot T^\varepsilon$  rational points of 'height' at most  $T$  in the 'transcendental' part of  $X$  (for the precise statement see Chapter 3).*

The Pila-Wilkie theorem shows that o-minimality has spectacular consequences for number theory. It provided a completely new proof strategy for problems in diophantine geometry and entailed many applications, including a famous reproof of the *Manin-Mumford conjecture* [34], and, recently, a proof of the *André-Oort conjecture* [35]. Thus, the Pila-Wilkie theorem is another evidence of the power of logical techniques applied in other areas of mathematics.

Inspired by the success of o-minimality and in the spirit of the introductory slogan, model theorists began looking for weaker tameness conditions on expansions of the real field beyond o-minimality. This started with Miller in 2005 who modified the definition of o-minimal structures in different ways and created the concept of *d-minimality* [26].

**Definition.** A structure  $\mathcal{R}$  expanding the real field is **d-minimal**, if in any elementarily equivalent structure, any definable subset of the universe is a finite union of discrete sets and open sets.

D-minimal structures inherit a lot of the theory of o-minimal geometry, establishing d-minimality as "the next best thing to o-minimality" (Miller, [27]). Examples of d-minimal structures that are not o-minimal, include the expansion of the real field by a predicate for the powers of 2 or by a predicate for the logarithmic spiral  $S_\omega := \{(e^t \cos(\omega t), e^t \sin(\omega t)) \mid t \in \mathbb{R}\}$  for some  $\omega \in \mathbb{R} \setminus \{0\}$ .

Since the concept of d-minimality is rather young, still a lot of progress can be expected from examining which properties from o-minimal geometry carry over (in adapted form) to the d-minimal setting. The starting question for the present thesis is the following:

May the Pila-Wilkie theorem be generalized to d-minimal structures? (\*)

Some research has been done so far on extending the Pila-Wilkie theorem to non-o-minimal structures. Eleftheriou showed that it can be extended to expansions of o-minimal structures by certain dense sets [17]. It is also easy to check that the bound from the theorem holds for the set  $2^{\mathbb{Z}}$ . Comte and Miller showed that it holds for the logarithmic spiral, too [8]. However, it is unclear whether it holds for all definable sets in the expansions of the real field generated by these sets, or, more general, for all definable sets in any d-minimal structure. Until now, no counterexample has been found, but the original proof does not work easily, either.

A final answer to (\*) could not be given in the context of the master's thesis. Instead, we focus on one of the main ingredients of the proof of the Pila-Wilkie theorem, the *parametrization theorem*. This theorem states that any definable set in an o-minimal structure can be parametrized by finitely many smooth maps that are not too oscillatory.

By closely examining the theorem and its proof, focusing on how to adapt it to the d-minimal setting, and exploiting the existing theory on d-minimality, a d-minimal version of the parametrization theorem was found.

**Theorem** (Countable Parametrization Theorem). *Work in a d-minimal expansion of the real field. Let  $k \in \mathbb{N}$ . Then any bounded definable set admits a 'countable k-parametrization' (for the precise statement, see Chapter 3).*

The structure of the thesis takes the following form: In the first chapter, the concept of o-minimality is introduced briefly. The major theorems are presented alongside the most important examples.

Chapter 2 deals with d-minimal structures. After presenting examples and d-minimal versions of the theorems from o-minimality, we elaborate on different notions of dimension and their relationship in d-minimal structures. This allows for deducing some metric properties of 0-dimensional definable subsets of the real line. Finally, we prove a d-minimal version of the uniform finiteness lemma from o-minimality under certain additional assumptions.

The third chapter is concerned with our main result, the countable parametrization theorem. First we state and discuss the Pila-Wilkie theorem and the parametrization theorem, before presenting the inductive proof of countable parametrization. After commenting on family versions of the theorem we conclude by giving an outlook on what is still missing on the way towards a d-minimal Pila-Wilkie theorem.

The field of o-minimality and d-minimality is located at the boundary of logic and geometry. Hence, proofs often require a mixture of model-theoretic and analytic-geometric techniques. This also holds for the present thesis. A basic understanding of the fundamental model-theoretic concepts such as structures and definable sets is assumed.

## Notations and Conventions

If not declared otherwise,  $m$ ,  $n$  and  $k$  denote arbitrary natural numbers.  $0$  is a natural number,  $\mathbb{N}_+$  denotes the set of positive integers,  $\mathbb{Z}_{<0}$  the negative integers. By an interval in a dense linear order, we always mean an open interval, possibly infinite. Put  $I := (0, 1)$ , the unit interval in  $\mathbb{R}$ . Put  $\mathbb{R}_+ := (0, \infty)$ .

'Definable' always means 'definable with parameters'. For structures, we use the calligraphic letters  $\mathcal{R}$  and  $\mathcal{M}$ . The corresponding universe is referred to by the Latin letters  $R$  and  $M$ . The *real line* is the structure  $(\mathbb{R}, <)$ . The structure of the *real field* is denoted by  $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot)$ . Note that  $\overline{\mathbb{R}}$  and  $(\mathbb{R}, <, +, \cdot)$  are interdefinable as the order is already definable in  $\overline{\mathbb{R}}$ .

For  $P \subset \mathbb{R}^m$  a parameter space,  $A \subset P \times \mathbb{R}^n$ , denote the projections by  $\pi_1 : A \rightarrow P$  and  $\pi_2 : A \rightarrow \mathbb{R}^n$ , if not declared otherwise. For  $x \in P$  a *parameter*, put

$$A_x := \{y \in \mathbb{R}^n \mid (x, y) \in A\},$$

the *fiber* of  $A$  over  $x$  (possibly empty). By a *definable family of sets* with parameters from  $P$  we mean a definable subset  $A \subset P \times \mathbb{R}^n$  regarded as a collection of its fibers. We also write  $A = \{A_x\}_{x \in P}$ . Similarly, a *definable family of maps* with parameters from  $P$  is a definable map  $f : A \rightarrow \mathbb{R}^k$  with  $A \subset P \times \mathbb{R}^n$  which we regard as a collection of maps  $f_x := f(x, -) : A_x \rightarrow \mathbb{R}^k$  for  $x \in P$  (possibly empty). We also write  $f = \{f_x : A_x \rightarrow \mathbb{R}^k\}_{x \in P}$ . If for a definable family of sets or maps something is bounded for each fiber, we say it is *uniformly bounded* if there is a bound not depending on the parameter.

For  $X$  a topological space,  $A \subset X$ , denote

- $\lim(A)$ , the set of limit points of  $A$
- $\text{isol}(A)$ , the set of isolated points of  $A$
- $\text{int}(A)$ , the interior of  $A$
- $\text{cl}(A)$ , the closure of  $A$
- $\text{fr}(A)$ , the frontier of  $A$  ( $= \text{cl}(A) \setminus A$ ).

Note that if  $X = \mathbb{R}^n$  and  $A$  is definable in a structure expanding  $(\mathbb{R}, <, +)$ , then all the above sets are definable.  $A$  is *discrete* if  $A = \text{isol}(A)$ .  $A$  is *nowhere dense* if the closure of  $A$  has no interior.

For  $f : X \rightarrow \mathbb{R}^n$  a sufficiently smooth function with  $X \subset \mathbb{R}^m$  open and  $\mu \in \mathbb{N}^m$  a multiindex, denote by  $f^{(\mu)}$  the partial derivative of  $f$  in direction  $\mu$ . By  $|\mu|$  denote the sum of the entries of  $\mu$ , equivalently, the order of the derivative  $f^{(\mu)}$ . If  $X \subset \mathbb{R}$ , we also write  $f'$  for  $f^{(1)}$ .

# Chapter 1

## O-minimality

In 1984, Grothendieck submitted a famous proposal for long-term mathematical research, the *Esquisse d'un Programme* [23]. Among other things, he formulated his vision of a *topologie modérée*, a *tame topology*. In classical geometry there exist many 'pathological' objects or counterintuitive results, including, for example, space-filling curves or Cantor-like sets. The vague expression 'tame' refers to a topology where such 'wild' phenomena are avoided. Accordingly, Grothendieck called for a new foundation of topology and geometry.

O-minimality can be seen as a realization of this hoped-for tame topology using an axiomatic model-theoretic approach. The concept is partly motivated by the study of *semialgebraic sets* and can be seen as a generalization of this. Semialgebraic sets are subsets of  $\mathbb{R}^n$  essentially defined by polynomial equalities and inequalities. Semialgebraic sets and their many nice properties have been well-known for a long time and they are a central object of study in real algebraic geometry as they naturally arise as projections of real algebraic sets. But their study is also connected to model theory since, by the Tarski-Seidenberg theorem, semialgebraic sets are exactly the sets definable in the structure of the real field. This opens the door to the question whether there are other structures sharing the tame properties of semialgebraic sets, leading to the concept of o-minimality.

A structure expanding a dense linear order is called *o-minimal* if every subset of its domain is already definable using only the order, or, equivalently, is a finite union of points and intervals. Surprisingly, this condition on subsets of the domain has deep consequences for all definable sets, including the important cell decomposition theorem discussed in this chapter.

A special focus in research is put on o-minimal expansions of the real field as this is the case classical geometry is interested in. In general, expansions of the real field can show very 'wild' behaviour. Consider the structure  $(\overline{\mathbb{R}}, \mathbb{Z})$ : The definable sets turn out to be already all the projective sets from descriptive set theory, i.e. the sets that can be obtained from Borel sets by repeated application of projection and taking complements. These sets can get arbitrarily complex and follow no geometric intuition; Cantor-like sets and space-filling curves being only the tip of the iceberg of 'wild' phenomena. Accordingly, the study of this structure is less the task of geometry but that of descriptive set theory. O-minimal expansions of the real field instead don't define the integers (by definition) and there are lots of o-minimal structures worth studying. Structures that are not o-minimal, but still do not define the integers, are topic of the next chapter.



O-minimal structures were first considered by van den Dries in 1984 in [10] and received the name 'o-minimal' in the same year by Pillay and Steinhorn [36]. The term was introduced in analogy to the concept of 'strongly minimal' structures from model theory. By now, a lot of theory has been developed for o-minimal structures in analogy to the semialgebraic case, proving that o-minimal structures are tame in many senses. The other direction of research is concerned with finding new structures that are o-minimal, thereby ensuring that the concept is not too restrictive (for example, Wilkie proved the structure  $(\overline{\mathbb{R}}, \text{exp})$  to be o-minimal [41]).

In this chapter we first present the definition of o-minimality and the most important examples. In the second section we summarize major results for o-minimal structures, following the fundamental textbook [13] by van den Dries.

## 1.1 Definition and examples

Although the most important o-minimal structures are structures on the real numbers, the property can be defined more generally for expansions of dense linear orders without endpoints:

**Definition 1.1.1.** A structure  $\mathcal{R}$  expanding a dense linear order without endpoints is **o-minimal**, if every definable subset of  $R$  is a finite union of intervals and points.

**Remark 1.1.2.** The finite unions of intervals and points are precisely the subsets of  $R$  that can already be defined using only the order on  $R$ . This is the reason for the name o-minimal, the 'o' standing for 'order'. The definition resembles the well-known notion of 'strongly minimal' structures where each definable subset of the domain is finite or cofinite, hence already definable without using any function or relation symbol.

O-minimality has its origin in the study of semialgebraic sets, so this is our first example.

**Definition 1.1.3.** A **semialgebraic** set in  $\mathbb{R}^n$  is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}$$

for some  $k \in \mathbb{N}$ ,  $f, g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_n]$ .

It is easy to see that the semialgebraic sets are precisely the quantifier-free definable sets in the structure  $(\mathbb{R}, <, +, \cdot)$ . But more is true: The Tarski-Seidenberg theorem states that the projection of a semialgebraic subset of  $\mathbb{R}^{n+1}$  on the first  $n$  coordinates is again semialgebraic. This is equivalent to saying that  $(\mathbb{R}, <, +, \cdot)$  admits quantifier elimination and implies that the semialgebraic sets are indeed all the definable sets in  $(\mathbb{R}, <, +, \cdot)$  (and hence also the definable sets in  $\overline{\mathbb{R}}$ , as  $\overline{\mathbb{R}}$  and  $(\mathbb{R}, <, +, \cdot)$  are interdefinable). Now the semialgebraic subsets of  $\mathbb{R}$  certainly are finite unions of points and intervals, yielding

**Theorem 1.1.4.** *The structure  $\overline{\mathbb{R}}$  is o-minimal.*

By now, there are numerous structures that have been proven to be o-minimal. Often the proofs proceed by showing (a weaker form of) quantifier elimination, possibly in an extended language.

**Examples 1.1.5.** (i) Call  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a **restricted analytic function**, if  $f$  is given by a fixed convergent power series inside the unit cube  $[-1, 1]^n$  and vanishes outside  $[-1, 1]^n$ . Let  $\mathbb{R}_{\text{an}}$  be the structure obtained from the real field together with all restricted analytic functions. Then  $\mathbb{R}_{\text{an}}$  is o-minimal (proven by van den Dries [9,

12]). Its theory is model complete and admits quantifier elimination in a slightly extended language. The definable sets are the so-called **globally subanalytic sets**, a subclass of the subanalytic sets known from real analytic geometry.

- (ii) The structure  $(\overline{\mathbb{R}}, \text{exp})$  is o-minimal. This major result was proven by Wilkie in 1991 [41], by showing that the structure is model complete, which can be seen as a weak form of quantifier elimination. The definable sets in this structure are precisely the **subexponential sets**, i.e. projections of zero sets of functions from the ring  $\mathbb{Z}[x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}]$  for some  $n \in \mathbb{N}$ .
- (iii) Even the structure  $(\mathbb{R}_{\text{an}}, \text{exp})$  that arises from merging the previous examples is o-minimal, shown in 1994 by van den Dries, Macintyre, Marker and Miller [15, 16]. Again, quantifier elimination in an extended language plays a crucial role.

**Remark 1.1.6.** One cannot permit any global analytic function. For example, the structure  $(\overline{\mathbb{R}}, \text{sin})$  is not o-minimal: The zero set of the sine function is not a finite union of points and intervals. Instead, this structure defines the integers.

**Remark 1.1.7.** Rolin, Speissegger and Wilkie proved that there is no largest o-minimal expansion of the real field, i.e. an o-minimal expansion that expands any other o-minimal expansion of the real field [37].

## 1.2 Theory

A lot of theory has been developed for o-minimal structures by now, showing that they are tame in many senses. In this section we summarize the most important results. All the proofs and further results (including the triangulization theorem and trivialization theorem) can be found in [13]. For the rest of this section, fix an o-minimal structure  $\mathcal{R}$  expanding a dense linear order  $(R, <)$  without endpoints.

We start with the monotonicity theorem, a powerful characterization of univariate definable functions:

**Theorem 1.2.1** (Monotonicity Theorem). *Let  $f : (a, b) \rightarrow R$  be a definable function on a (possibly infinite) interval in  $R$ . Then there exist  $n \in \mathbb{N}$  and  $a_1 < \dots < a_n$  in  $(a, b)$  s.t. on each subinterval  $(a_i, a_{i+1})$  (for  $i = 0, \dots, n$ , with  $a_0 = a$  and  $a_{n+1} = b$ ),  $f$  is either constant, or strictly monotone and continuous.*

We continue with the main structure theorem for definable sets in o-minimal structures: the cell decomposition theorem. It states that any definable set can be partitioned into finitely many cells - particularly simple definable sets. We need some definitions to make this precise. For  $X \subset R^n$  denote

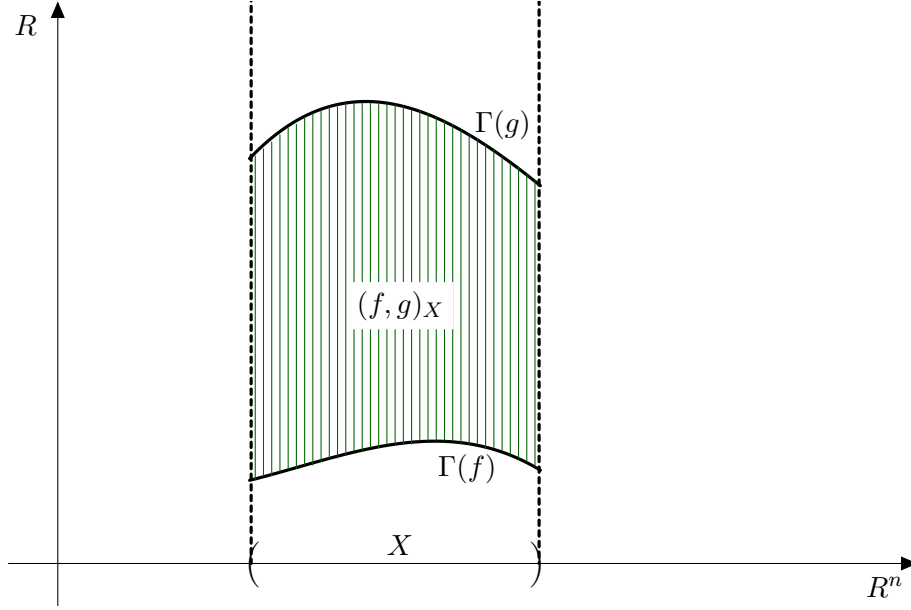
$$C(X) := \{f : X \rightarrow R \mid f \text{ is definable and continuous}\} \quad \text{and}$$

$$C_\infty(X) := C(X) \cup \{\infty, -\infty\},$$

where  $\infty$  and  $-\infty$  are regarded as constant functions on  $X$ . For  $f, g \in C_\infty(X)$  with  $f < g$  (meaning  $f(x) < g(x)$  for all  $x \in X$ ), put

$$(f, g)_X := \{(x, y) \in X \times R \mid f(x) < y < g(x)\},$$

a definable subset of  $R^{n+1}$ .



**Definition 1.2.2.** Let  $(i_1, \dots, i_n)$  be a sequence of zeros and ones of length  $n \in \mathbb{N}$ . Define an  $(i_1, \dots, i_n)$ -cell, a definable subset of  $R^n$ , recursively as follows:

- $R^0$  is the unique  $()$ -cell. A  $(0)$ -cell is a one-element subset of  $R$ , a  $(1)$ -cell is an interval in  $R$  (possibly infinite).
- An  $(i_1, \dots, i_n, 0)$ -cell is the graph  $\Gamma(f)$  of a function  $f \in C(X)$ , where  $X$  is an  $(i_1, \dots, i_n)$ -cell.
- An  $(i_1, \dots, i_n, 1)$ -cell is a set  $(f, g)_X$ , where  $X$  is an  $(i_1, \dots, i_n)$ -cell and  $f, g \in C_\infty(X)$ ,  $f < g$ .

A cell in  $R^n$  is an  $(i_1, \dots, i_n)$ -cell for some (necessarily unique) sequence  $(i_1, \dots, i_n)$ .

**Definition 1.2.3.** A decomposition of  $R^n$  is recursively defined as follows:

- $\{R^0\}$  is the unique decomposition of  $R^0$ .
- A decomposition of  $R^{n+1}$  is a partition  $\mathcal{D}$  of  $R^{n+1}$  into finitely many cells such that  $\pi\mathcal{D} := \{\pi C \mid C \in \mathcal{D}\}$  is a decomposition of  $R^n$ , where  $\pi : R^{n+1} \rightarrow R^n$  is the projection on the first  $n$  coordinates.

**Definition 1.2.4.** A collection  $\mathcal{A}$  of subsets of  $\mathbb{R}^n$  is **compatible** with a collection  $\mathcal{B}$  of subsets of  $\mathbb{R}^n$ , if for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ , either  $A$  is contained in  $B$  or  $A$  is disjoint from  $B$ .

Now we are able to state cell decomposition:

**Theorem 1.2.5** (Cell Decomposition). *Given any finite collection  $\mathcal{A}$  of definable subsets of  $R^n$ , there exists a decomposition of  $R^n$  compatible with  $\mathcal{A}$ .*

*For each definable function  $f : X \rightarrow R$ ,  $X \subset R^n$ , there is a decomposition  $\mathcal{D}$  of  $R^n$  compatible with  $\{X\}$  such that for each cell  $C \in \mathcal{D}$  with  $C \subset X$ , the restriction  $f|_C$  is continuous.*

**Remark 1.2.6.** Cell decomposition illustrates how a condition on definable subsets of  $R$  can have deep consequences for definable subsets of any  $R^n$ . In particular, the defining property of having finitely many connected components carries over from definable subsets of  $R$  to arbitrary definable sets.

The proof of cell decomposition makes use of the following theorem about definable families of finite sets:

**Theorem 1.2.7** (Uniform Finiteness). *Let  $A \subset R^m \times R$  be a definable family of finite subsets of  $R$ . Then there exists  $N \in \mathbb{N}$  s.t.  $\#A_x \leq N$  for all  $x \in R^m$ .*

O-minimal structures allow for a well-behaved notion of dimension:

**Definition 1.2.8.** For a definable set  $X \subset R^n$  define its **dimension**  $\dim X$  as follows:

$$\dim X = \max\{i_1 + \dots + i_n \mid X \text{ contains an } (i_1, \dots, i_n)\text{-cell}\}$$

and put  $\dim \emptyset = -\infty$ .

Exemplarily we mention two nice properties of  $\dim$ .

**Theorem 1.2.9.** *Let  $X \subset R^n$  be definable and  $f : X \rightarrow R^m$  a definable map. Then  $\dim X \geq \dim f(X)$ .*

Note that this excludes the existence of definable space-filling curves - a major problem in non-tame settings.

**Theorem 1.2.10.** *Let  $X \subset R^n$  be definable nonempty. Then  $\dim \text{fr } X < \dim X$ .*

As a corollary, van den Dries in [13] deduces that each closed definable set admits a stratification - another requirement for a 'tame' topology as formulated by Grothendieck [23].

The theory becomes even richer if more structure is available. From now on let  $\mathcal{R}$  additionally expand an ordered group  $(R, <, +)$ . One can show that the group then needs to be abelian and divisible, so the group structure allows for definably selecting the midpoint of an interval  $(a, b)$  by  $\frac{a+b}{2}$ . This is the basis for

**Theorem 1.2.11** (Definable Choice).

- (Definable Skolem Functions) *Let  $A \subset R^{m+n}$  be definable and  $\pi : R^{m+n} \rightarrow R^m$  the projection on the first  $m$  coordinates. Then there is a definable map  $f : \pi A \rightarrow R^n$  s.t.  $\Gamma(f) \subset A$ .*
- (Elimination of Imaginaries) *Each definable equivalence relation on a definable set  $A$  has a definable set of representatives.*

For the rest of this section let  $\mathcal{R}$  even expand an ordered field. In this case, one can (in analogy to the case  $R = \mathbb{R}$ ) define differentiability and continuous differentiability in  $R$ . A lot of real analysis then carries over to the general o-minimal setting (for a thorough treatment of this see [13, Chapter 7]). Exemplarily we mention the mean value theorem:

**Theorem 1.2.12** (Mean Value Theorem). *Let  $a < b$  in  $R$  and  $f : [a, b] \rightarrow R$  definable, continuous, differentiable on  $(a, b)$  with  $f(a) = f(b)$ . Then  $f'(c) = 0$  for some  $c \in (a, b)$ .*

Important is the following  $C^k$ -version of cell decomposition. For  $k \in \mathbb{N}$ , define  $C^k$ -cells in the same way as cells, but requiring all the functions appearing in the definition to be  $C^k$ -smooth. A  $C^k$ -**decomposition** is a decomposition consisting of  $C^k$ -cells. Then we have:

**Theorem 1.2.13** ( $C^k$ -Cell Decomposition). *Let  $k \in \mathbb{N}$ . Given any finite collection  $\mathcal{A}$  of definable subsets of  $R^n$ , there exists a  $C^k$ -decomposition of  $R^n$  compatible with  $\mathcal{A}$ .*

*For each definable function  $f : X \rightarrow R$ ,  $X \subset R^n$ , there is a  $C^k$ -decomposition  $\mathcal{D}$  of  $R^n$  compatible with  $\{X\}$  such that for each cell  $\mathcal{C} \in \mathcal{D}$  with  $\mathcal{C} \subset X$ , the restriction  $f|_{\mathcal{C}}$  is  $C^k$ -smooth.*

The second part of the theorem is also referred to as 'generic smoothness'.

## Chapter 2

# D-minimality

In the last chapter we have seen that o-minimal structures allow for a particularly well-behaved, tame geometry. In contrast, expansions of the real field that define the integers are highly complicated and full of non-tame phenomena such as space-filling curves. The question arising naturally is: What happens in between? Are there suitable conditions on structures, weaker than o-minimality, yet also ensuring tameness in some reasonable sense? In Miller's fundamental paper *Tameness in Expansions of the Real Field* from 2005 [26] he puts it as follows:

"What might it mean for a first-order expansion of the field of real numbers to be tame or well behaved? In recent years, much attention has been paid by model theorists and real-analytic geometers to the o-minimal setting [...]. But there are expansions of the real field that define sets with infinitely many connected components, yet are tame in some well-defined sense [...]. The analysis of such structures often requires a mixture of model-theoretic, analytic-geometric and descriptive set-theoretic techniques. An underlying idea is that first-order definability, in combination with the field structure, can be used as a tool for determining how complicated is a given set of real numbers."

This has become the starting point of a whole research program dealing with tameness beyond o-minimality.

An expansion  $\mathcal{R}$  of the real line is o-minimal, if any definable subset of  $\mathbb{R}$  either has interior or is finite. An intuitive way to create weaker conditions on theories that potentially still imply some sort of tameness, is to loosen this definition by replacing 'finite' by some weaker condition. Miller examines various such conditions, among others he replaces 'finite' by 'finite union of discrete sets' - the birth of *d-minimality* ('d' standing for 'discrete'). A lot of results for o-minimal structures turned out to hold in an adapted form also for d-minimal ones, establishing d-minimality as "the next best thing to o-minimality" [27]. Parallel to this, a lot of non-o-minimal structures have been proven to be d-minimal, justifying the relevance of the concept.

In this chapter we will first get to know equivalent definitions of d-minimality and introduce the Cantor-Bendixson rank as well as the notion of *noiseless* structures. After seeing some examples of d-minimal structures, a general theory of d-minimality is developed consisting mostly of adaptations of the theorems from o-minimality presented in the last chapter. Section 2.4 is concerned with the relationship of different notions of dimension in noiseless structures. In the last two sections we prove some new results: Section 2.5

analyzes metric properties of 0-dimensional subsets of  $\mathbb{R}$  and convergence rates of definable sequences. The underlying idea is that the points cannot lie evenly distributed over the real line. In the last section we show a d-minimal version of uniform finiteness for structures with good control over the convergence rate of definable sequences.

## 2.1 Definition and Cantor-Bendixson rank

**Definition 2.1.1.** An expansion  $\mathcal{R}$  of  $(\mathbb{R}, <)$  is called **d-minimal**, if for every  $\mathcal{M} \equiv \mathcal{R}$ , every subset of  $M$  definable in  $\mathcal{M}$  is the union of an open set and finitely many discrete sets.

Note that one could replace 'being the union of an open set and finitely many discrete set' by 'having interior or being a finite union of discrete sets' for an equivalent definition.

**Remark 2.1.2.** The notion of d-minimality can be generalized to structures  $\mathcal{R}$  extending some dense linear order without endpoints. Then one additionally requires  $R$  to be **definably connected**, i.e. to have no proper non-empty definable clopen subset ( $\mathbb{R}$  is connected, thus definably connected). However, in the following we will focus on structures expanding the real line.

**Remark 2.1.3.** D-minimality is preserved under elementary equivalence by definition, other than o-minimality where this result requires a non-trivial proof. It is still open whether this requirement is necessary. There is no example known so far of an expansion of the real field with every subset of  $\mathbb{R}$  having interior or being a finite union of discrete sets s.t. this property fails in some elementarily equivalent structure.

In order to get a better intuition for what it means for a set to be a finite union of discrete sets we introduce the Cantor-Bendixson rank.

**Definition 2.1.4.** Let  $X$  be a topological space,  $A \subset X$ . For ordinals  $\lambda$ , define the  **$\lambda$ -th Cantor-Bendixson derivative**  $A^{[\lambda]}$  as follows:

$$\begin{aligned} A^{[0]} &= A \\ A^{[\lambda+1]} &= A^{[\lambda]} \setminus \text{isol}(A^{[\lambda]}) && (= \text{lim}(A^{[\lambda]}) \cap A^{[\lambda]}) \\ A^{[\lambda]} &= A \setminus \bigcup_{\mu < \lambda} \text{isol}(A^{[\mu]}) && \text{if } \lambda \text{ is a limit} \end{aligned}$$

This defines a  $\subset$ -decreasing transfinite sequence of sets that must eventually be constant. The **Cantor-Bendixson rank** of  $A$  is the smallest ordinal  $\lambda$  s.t.  $A^{[\lambda]} = \emptyset$ . If no such ordinal exists, it is put to  $\infty$ .

The connection of Cantor-Bendixson rank and d-minimality lies in the following proposition:

**Proposition 2.1.5.** *Let  $N \in \mathbb{N}$ .  $A$  is the union of an open set and  $N$  discrete sets if and only if  $A \setminus \text{int}(A)$  has Cantor-Bendixson rank  $\leq N$ .*

*Proof.* The proof is partly based on [20, §1].

$\Rightarrow$ : We proceed by induction on  $N$ . The base case  $N = 0$  is clear. Assume the claim holds for  $N \in \mathbb{N}$  and let  $A$  be the union of an open and  $N + 1$  discrete sets, then  $B := A \setminus \text{int} A$  is the union of  $N + 1$  discrete sets, say  $B_1, \dots, B_{N+1}$ . We have to show that the Cantor-Bendixson rank of  $B$  is  $\leq N + 1$ .

Define  $C_i := \lim(B_i) \cap B$  (closed in  $B$  as the set of limit points is always closed). Note that taking limit points commutes with finite unions, implying

$$B^{[1]} = \lim(B) \cap B = \bigcup_{i=1}^{N+1} \lim(B_i) \cap B = \bigcup_{i=1}^{N+1} C_i$$

**Claim.** For all  $k \in \mathbb{N}$  we have  $(\bigcup_{i=1}^{N+1} C_i)^{[k]} = \bigcup_{i=1}^{N+1} C_i^{[k]}$ .

*Proof of Claim.* We prove this for  $k = 1$ , only using that  $C_i$  is closed in  $B$ . This property carries over to all the  $C_i^{[k]}$  as  $C_i^{[k]}$  is closed in  $C_i^{[k-1]}$  by definition. The claim then follows inductively. Note that

$$\lim(C_i) \cap \bigcup_{j=1}^{N+1} C_j = \lim(C_i) \cap C_i.$$

For, if  $x$  is a limit point of  $C_i$  inside  $\bigcup_{j=1}^{N+1} C_j \subset B$ , it must already lie in  $C_i$  as  $C_i$  is closed in  $B$ . Now

$$\left(\bigcup_{i=1}^{N+1} C_i\right)^{[1]} = \bigcup_{i=1}^{N+1} \lim(C_i) \cap \bigcup_{j=1}^{N+1} C_j = \bigcup_{i=1}^{N+1} \lim(C_i) \cap C_i = \bigcup_{i=1}^{N+1} C_i^{[1]}.$$

□

The claim implies that the Cantor-Bendixson rank of  $B^{[1]}$  equals the maximal Cantor-Bendixson rank of the  $C_i$ , provided that this is finite. Now for each  $i = 1, \dots, N + 1$

$$C_i = \lim(B_i) \cap B = \bigcup_{j=1}^{N+1} \lim(B_i) \cap B_j.$$

As  $B_i$  is discrete,  $\lim(B_i) \cap B_i = \emptyset$ , hence  $C_i$  is the union of only  $N$  discrete sets. By the inductive assumption, its Cantor-Bendixson rank is  $\leq N$ . Thus, the Cantor-Bendixson rank of  $B^{[1]}$  is  $\leq N$ , equivalently, the Cantor-Bendixson rank of  $B$  is  $\leq N + 1$ .

$\Leftarrow$ : Let the Cantor-Bendixson rank of  $B := A \setminus \text{int}(A)$  be  $\leq N$ , then  $B^{[N]} = \emptyset$ . Thus,

$$A = \text{int } A \cup \left(\bigcup_{i=1}^N B^{[i-1]} \setminus B^{[i]}\right).$$

But  $B^{[i-1]} \setminus B^{[i]} = \text{isol}(B^{[i-1]})$  is discrete for all  $i = 1, \dots, N$ . □

**Remark 2.1.6.** If  $A \subset \mathbb{R}$  is definable in a structure expanding the real line, then all the  $A^{[i]}$  are definable for  $i \in \mathbb{N}$ . Thus, by Proposition 2.1.5, the sentence ' $A$  is a finite union of at most  $N$  discrete sets' can be expressed in first order language by stating that  $A^{[N]} = \emptyset$ .

We continue by giving several equivalent criteria for d-minimality. In contrast to the definition of o-minimality, the condition on definable subsets of  $\mathbb{R}$  for d-minimal structures is required to hold in all elementary equivalent structures. This is equivalent to requiring that the number of discrete sets needed to cover the non-interior part is uniformly bounded for definable families of subsets of  $\mathbb{R}$  (Condition (2)). The open set and the discrete sets can even be chosen to form definable families themselves (Condition (4)).

Finally, being the union of finitely many discrete sets is equivalent to having finite Cantor-Bendixson rank, so we can reformulate the definition accordingly (Condition (3)).



**Proposition 2.1.7.** *Let  $\mathcal{R}$  be an expansion of  $(\mathbb{R}, <)$ . The following are equivalent:*

- (1)  $\mathcal{R}$  is d-minimal.
- (2) For every  $m \in \mathbb{N}$  and definable  $A \subset \mathbb{R}^m \times \mathbb{R}$  there exists  $N \in \mathbb{N}$  such that for every  $x \in \mathbb{R}^m$ ,  $A_x$  either has interior or is a union of at most  $N$  discrete sets.
- (3) For every  $m \in \mathbb{N}$  and definable  $A \subset \mathbb{R}^{m+1}$  there exists  $N \in \mathbb{N}$  such that for every  $x \in \mathbb{R}^m$ ,  $A_x \setminus \text{int}(A_x)$  has Cantor-Bendixson rank  $\leq N$ .
- (4) For every  $m \in \mathbb{N}$  and definable  $A \subset \mathbb{R}^m \times \mathbb{R}$  there exists a definable family of open sets  $O \subset \mathbb{R}^m \times \mathbb{R}$  and  $N \in \mathbb{N}$  and definable families of discrete sets  $D_1, \dots, D_N \subset \mathbb{R}^m \times \mathbb{R}$  s. t.  $A = O \cup D_1 \cup \dots \cup D_N$ .

*Proof.* The proof is partly based on [26, §8.5].

(1)  $\Rightarrow$  (2): Assume the contrary, i.e. there is no uniform bound on the number of discrete sets needed to cover the fibers of  $A$  without interior. Replacing the parameters in the defining formula for  $A$  by variables, we obtain  $n \in \mathbb{N}$  and  $B \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$  a  $\emptyset$ -definable set s.t.  $A = B_z$  for some  $z \in \mathbb{R}^n$ . Consider the following set of formulas in the free variables  $z, x$  (to put this in first order language, use the Cantor-Bendixson derivative as mentioned above):

$$\{B_{z,x} \text{ does not have interior and is not the union of } N \text{ discrete sets}\}_{N \in \mathbb{N}}$$

This is finitely satisfiable by assumption, hence forms an  $(n+m)$ -type. By compactness, there is an elementary extension of  $\mathcal{R}$  realizing it. This is impossible by (1).

(2)  $\Rightarrow$  (1): Let  $\mathcal{M} \equiv \mathcal{R}$  and  $S \subset M$  definable in  $\mathcal{M}$ , say by a formula  $\varphi(a, y)$  with  $a \in M^m$ ,  $y$  a variable. By (2) there is  $N \in \mathbb{N}$ , s.t.  $\mathcal{R}$  satisfies the sentence

'For all fixed  $x \in \mathbb{R}^m$ , the set defined by  $\varphi(x, y)$  either has interior or is the union of at most  $N$  discrete sets!'

This is a sentence in first-order language (use the Cantor-Bendixson derivative as above). Since  $\mathcal{M} \equiv \mathcal{R}$ , also  $\mathcal{M}$  satisfies the sentence implying that  $S$  either has interior or is a finite union of discrete sets.

(2)  $\Leftrightarrow$  (3): immediate by Proposition 2.1.5

(3)  $\Rightarrow$  (4): Define  $O = \{\text{int } A_x\}_{x \in \mathbb{R}^m}$  and  $D_i = \{(A_x \setminus \text{int } A_x)^{[i-1]} \setminus (A_x \setminus \text{int } A_x)^{[i]}\}_{x \in \mathbb{R}^m}$ .

(4)  $\Rightarrow$  (2): obvious □

At the end of this section, we introduce another notion in between d-minimality and the chaos of the real projective hierarchy, that arises from replacing 'finite' in the definition of o-minimality by 'nowhere dense':

**Definition 2.1.8.** An expansion  $\mathcal{R}$  of  $(\mathbb{R}, <)$  is called **noiseless**<sup>1</sup>, if every subset of  $\mathbb{R}$  definable in  $\mathcal{R}$  either has interior or is nowhere dense.

Certainly, any d-minimal structure is noiseless. An example of a noiseless structure that is not d-minimal can be found in [21]. Some of the theory for d-minimal structures can be obtained by results about noiseless structures as we will see in Section 2.3.

<sup>1</sup>The term was introduced by Hieronymi, Nell and Walsberg [25] after a suggestion of Miller. The same property has been called *i-minimality* by Fornasiero [18].



## 2.2 Examples

In this section we present some important examples of d-minimal structures that are not o-minimal. More examples can be found in [19, 26, 29].

We start with a basic example that had already been found to be well-behaved by van den Dries in 1987 [11] before the concept of d-minimality was introduced.

**Example 2.2.1.** The structure  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$  is d-minimal. More generally, for any o-minimal expansion  $\mathcal{R}$  of  $\overline{\mathbb{R}}$  with **field of exponents**  $\mathbb{Q}$  (that is, only rational powers are definable) and  $\alpha \in \mathbb{R}_+$ ,  $(\mathcal{R}, \alpha^{\mathbb{Z}})$  is d-minimal.

*Proof.* [11, Theorem III] implies that any subset of  $\mathbb{R}$  definable in  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$  has interior or is the union of finitely many discrete sets. The general case is proven by Miller in [26], adapting methods from [11].  $\square$

Intuitively,  $2^{\mathbb{N}}$ , seen as an increasing sequence, grows fast enough to allow for tame behaviour. In contrast, expanding the real field by a sequence such as  $(i^n)_{i \in \mathbb{N}}$  for some  $n \in \mathbb{N}_+$ , that grows only polynomially, immediately defines the integers (as taking  $n$ -th roots is definable in  $\overline{\mathbb{R}}$ ). More generally, Friedman and Miller define fast sequences and show that one obtains d-minimal structures from them:

**Definition 2.2.2.** Let  $\mathcal{R}$  be a structure on  $\mathbb{R}$ ,  $(a_i)_{i \in \mathbb{N}}$  a strictly increasing sequence of positive real numbers. We say that  $(a_i)_{i \in \mathbb{N}}$  is  **$\mathcal{R}$ -fast**, if for each definable  $f : \mathbb{R} \rightarrow \mathbb{R}$  there exists  $i_0 \in \mathbb{N}$  s.t.  $f(a_i) < a_{i+1}$  for all  $i > i_0$ .

**Theorem 2.2.3** (Friedman and Miller, [20]). *If  $\mathcal{R}$  is an o-minimal expansion of  $(\mathbb{R}, <, +)$  and  $(a_i)_{i \in \mathbb{N}}$  an  $\mathcal{R}$ -fast sequence, then  $(\mathcal{R}, \{a_i \mid i \in \mathbb{N}\})$  is d-minimal.*

From such a d-minimal structure, that expands an o-minimal structure by a subset  $E$  of  $\mathbb{R}$  without interior, one obtains a new d-minimal structure by adding all the subsets of Cartesian powers of  $E$ :

**Definition 2.2.4.** Let  $\mathcal{R}$  be a structure on  $\mathbb{R}$ ,  $E \subset \mathbb{R}$  with  $\text{int } E = \emptyset$ . We denote by  $(\mathcal{R}, E)^{\#}$  the structure  $\mathcal{R}$  together with a predicate for every subset of every cartesian power of  $E$ .

**Theorem 2.2.5** (Friedman and Miller, [19, 26]). *Let  $\mathcal{R}$  be an o-minimal expansion of  $(\mathbb{R}, <, +)$ ,  $E \subset \mathbb{R}$  without interior s.t.  $(\mathcal{R}, E)$  is d-minimal. Then  $(\mathcal{R}, E)^{\#}$  is also d-minimal.*

As a corollary of Example 2.2.1, we obtain another important example of a d-minimal structure arising from the logarithmic spiral:

**Example 2.2.6.** For  $\omega \in \mathbb{R} \setminus \{0\}$  let  $S_\omega := \{(e^t \cos(\omega t), e^t \sin(\omega t)) \mid t \in \mathbb{R}\}$ , the logarithmic spiral. Then the structure  $(\overline{\mathbb{R}}, S_\omega)$  is d-minimal.

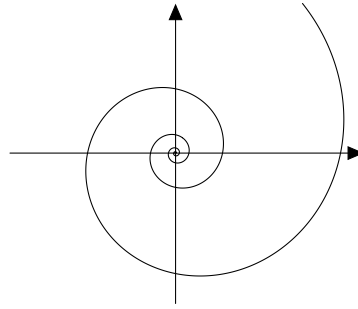
*Proof.* The proof is taken from [26].

The structure  $\mathbb{R}^{\text{RE}} := (\mathbb{R}, \exp|_{[0, 2\pi]}, \sin|_{[0, 2\pi]})$  is o-minimal with field of exponents  $\mathbb{Q}$  [12]. Note that  $\cos|_{[0, 2\pi]}$  is definable in  $\mathbb{R}^{\text{RE}}$ . Put  $\alpha = e^{2\pi/\omega}$ . Then

$$(x, y) \in S_\omega \Leftrightarrow \exists g \in \alpha^{\mathbb{Z}} \exists t \in [0, 2\pi/|\omega|] : x = ge^t \cos(\omega t), y = ge^t \sin(\omega t)$$

Hence,  $S_\omega$  is definable in  $(\mathbb{R}^{\text{RE}}, \alpha^{\mathbb{Z}})$  which is d-minimal by Example 2.2.1.  $\square$

**Remark 2.2.7.** Logarithmic spirals appear naturally as trajectories of solutions of certain differential equations. Miller shows that any expansion of the real field by certain locally closed trajectories of linear vector fields either is o-minimal, defines the integers, or equals  $(\overline{\mathbb{R}}, S_\omega)$  for some  $\omega \in \mathbb{R} \setminus \{0\}$  [28, Theorem 1.1]. This result underlines the importance of the concept of d-minimality.



Finally, we relate d-minimality to another notion that is weaker than o-minimality: *local o-minimality*, introduced by Toffalori and Vozoris [39].

**Definition 2.2.8.** Let  $\mathcal{M}$  be a structure expanding a dense linear order without endpoints.  $\mathcal{M}$  is called **locally o-minimal**, if for every  $a \in M$  and definable set  $A \subset M$ , there is an interval  $J$  around  $a$  s.t.  $A \cap J$  is a finite union of points and intervals.

**Remark 2.2.9.** Any o-minimal structure is locally o-minimal. Any locally o-minimal structure expanding the real field is o-minimal. However, there are locally o-minimal structures expanding the real line that are not o-minimal.

**Proposition 2.2.10.** *Every locally o-minimal expansion of  $(\mathbb{R}, <)$  is d-minimal.*

*Proof.* Let  $\mathcal{R}$  be a locally o-minimal structure expanding the real line. Let  $A \subset \mathbb{R}$  be definable in  $\mathcal{R}$ . Assume  $A$  has no interior. We show that  $A$  is discrete. Let  $a \in A$ . By local o-minimality, there exists an interval  $J$  around  $a$  s.t.  $A \cap J$  is a finite union of points and intervals, as  $A$  has no interior, it must be finite. Shrinking the interval, we can find a neighbourhood of  $a$  not containing any other element of  $A$ . Thus,  $A$  is discrete.

We have shown that any definable subset of  $\mathbb{R}$  has interior or is the union of finitely many discrete sets, with the number of discrete sets needed for the cover uniformly bounded by 1. Using Condition (2) from Proposition 2.1.7, this implies that  $\mathcal{R}$  is d-minimal.  $\square$

**Remark 2.2.11.** Not every d-minimal structure is locally o-minimal: The set  $2^{\mathbb{Z}}$  is definable in the d-minimal structure  $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}})$ , but locally at 0 not a finite union of points and intervals.

**Remark 2.2.12.** The lemma provides us a new class of examples of d-minimal structures that are not expansions of the real field. For example, the structure  $(\mathbb{R}, <, +, \sin)$  is locally o-minimal [39], hence also d-minimal.

## 2.3 Theory

Much of the theory for o-minimality carries over to the d-minimal setting. The general slogan is that what is 'finite set' in o-minimality has to be replaced by 'finite union of discrete sets' or weaker versions of smallness. The results for d-minimal structures include:

- D-minimal monotonicity theorem
- D-minimal generic smoothness
- D-minimal cell decomposition
- Definable choice

This outlines the section. Most proofs are omitted.

Similar to o-minimal ones, d-minimal structures also allow for a well-behaved notion of dimension studied in the next section together with some consequences. More properties, not only for d-minimality, but also weaker notions of tameness, can be found in [26].

The d-minimal monotonicity theorem and generic smoothness can be derived from the following structure theorem for noiseless structures:

**Theorem 2.3.1** (Miller, [26, Theorem 3.3]). *Let  $\mathcal{R}$  be a noiseless expansion of  $(\mathbb{R}, <)$ . Then every definable set has interior or is nowhere dense. If  $U \subset \mathbb{R}^m$  is open,  $f : U \rightarrow \mathbb{R}$  definable, then there is an open definable  $V \subset U$  such that  $U \setminus V$  is nowhere dense and the restriction  $f|_V : V \rightarrow \mathbb{R}$  is continuous. If  $m = 1$ ,  $V$  may be chosen so that for each connected component  $J$  of  $V$ ,  $f|_J$  is either constant or strictly monotone. If moreover  $\mathcal{R}$  expands  $\overline{\mathbb{R}}$ , the above holds with ' $C^k$ -smooth' instead of 'continuous' for any  $k \in \mathbb{N}$ .*

**Corollary 2.3.2** (D-minimal Monotonicity Theorem). *Let  $\mathcal{R}$  be a d-minimal expansion of  $(\mathbb{R}, <)$ ,  $U \subset \mathbb{R}$  open,  $f : U \rightarrow \mathbb{R}$  definable. Then there is  $V \subset U$  open definable such that  $U \setminus V$  is a finite union of discrete sets and for each connected component  $J$  of  $V$ ,  $f|_J$  is either constant or strictly monotone and continuous.*

*Proof.* This follows immediately from Theorem 2.3.1, noting that in d-minimal structures, definable nowhere dense subsets of  $\mathbb{R}$  must be finite unions of discrete sets.  $\square$

**Corollary 2.3.3** (D-minimal Generic Smoothness). *Let  $\mathcal{R}$  be a d-minimal expansion of the real line,  $U \subset \mathbb{R}^m$  open,  $f : U \rightarrow \mathbb{R}$  definable. Then there is an open definable  $V \subset U$  such that  $U \setminus V$  is nowhere dense and the restriction  $f|_V$  is continuous. If  $\mathcal{R}$  expands  $\overline{\mathbb{R}}$ , the above holds with ' $C^k$ -smooth' instead of 'continuous' for any  $k \in \mathbb{N}$ .*

**Remark 2.3.4.** Here we cannot replace 'nowhere dense' by 'finite union of discrete sets' as we don't have that arbitrary definable nowhere dense sets are finite union of discrete sets in d-minimal structures.

Now we turn towards cell decomposition. For d-minimal structures, the best version of cell decomposition currently at hand is countable cell decomposition.

**Definition 2.3.5.** Work in a d-minimal expansion of the real field. Define **cells** and  **$C^k$ -cells** exactly as in the o-minimal case (cf. Definition 1.2.2). Define a **countable decomposition** of  $\mathbb{R}^n$  by induction on  $n$ :

- $\{\mathbb{R}^0\}$  is the unique countable decomposition of  $\mathbb{R}^0$ .
- A countable decomposition of  $\mathbb{R}^{n+1}$  is a countable partition  $\mathcal{D}$  of  $\mathbb{R}^{n+1}$  into cells such that  $\pi\mathcal{D} := \{\pi\mathcal{C} \mid \mathcal{C} \in \mathcal{D}\}$  is a countable decomposition of  $\mathbb{R}^n$  (where  $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  is the projection on the first  $n$  coordinates).

For  $k \in \mathbb{N}$ , a **countable  $C^k$ -decomposition** is a countable decomposition consisting of  $C^k$ -cells.

**Theorem 2.3.6** (Countable Cell Decomposition, Miller, [26, Theorem 4]). <sup>2</sup> *Work in a d-minimal expansion of  $\overline{\mathbb{R}}$ . Given any finite collection  $\mathcal{A}$  of definable subsets of  $\mathbb{R}^n$ , for any  $k \in \mathbb{N}$  there exists a countable  $C^k$ -decomposition of  $\mathbb{R}^n$  compatible with  $\mathcal{A}$ .*

<sup>2</sup>An error in Miller's proof was repaired by Thamrongthanyalak [38].

This theorem will be crucial when proving the countable parametrization theorem in Chapter 3. However, the result does not seem to be optimal in the sense that it not fully captures the structure of definable sets in d-minimal expansions of the real field. For example, discrete subsets of  $\mathbb{R}$  definable in d-minimal expansions of  $\overline{\mathbb{R}}$ , such as  $2^{\mathbb{N}}$ , certainly are countable unions of points. But there is more to say, the points cannot be distributed evenly across the line (otherwise the integers would be definable), instead the sequence grows faster than polynomially. For countable cell decompositions of arbitrary definable sets one would expect some relation between the cells or their sizes and positions that determines the decomposition beyond being 'countable'. Until now, in this sense, no satisfying d-minimal version (or even more abstract version) of cell decomposition has been found.

In Chapter 3 we will need family versions of the d-minimal monotonicity theorem, generic smoothness and countable cell decomposition. These can be derived easily:

**Theorem 2.3.7** (Family Version of D-minimal Monotonicity Theorem). *Work in a d-minimal expansion of the real line. Let  $U = \{U_s\}_{s \in \mathbb{R}^m}$  be a definable family of open subsets of  $\mathbb{R}$ ,  $f = \{f_s : U_s \rightarrow \mathbb{R}\}_{s \in \mathbb{R}^m}$  a definable family of functions. Then there is  $V \subset U$  a definable family of open subsets such that for each  $s \in \mathbb{R}^m$ ,  $U_s \setminus V_s$  is a finite union of discrete sets and for each connected component  $J$  of  $V_s$ ,  $(f_s)|_J$  is either constant or strictly monotone and continuous.*

*Proof.* Say  $f_s$  is constant (strictly monotone and continuous) at  $x \in U_s$  if there is  $r \in \mathbb{R}_+$  s.t.  $f_s$  is constant (strictly monotone and continuous) on  $(x - r, x + r)$ . Now, for  $s \in \mathbb{R}^m$ , set

$$V_s := \{x \in U_s \mid f_s \text{ is constant or strictly monotone at } x\}.$$

Then  $V = \{V_s\}_{s \in \mathbb{R}^m}$  is definable and  $V_s$  is open. Moreover, by definition of  $V_s$ ,  $(f_s)|_J$  is either constant or strictly monotone and continuous for each connected component  $J$  of  $V_s$ . Left to show is that  $U_s \setminus V_s$  is countable. But this follows from the d-minimal monotonicity theorem applied to  $f_s$ .  $\square$

**Theorem 2.3.8** (Family Version of D-minimal Generic Smoothness). *Work in a d-minimal structure expanding the real field. Let  $U = \{U_s\}_{s \in \mathbb{R}^m}$  be a definable family of open subsets of  $\mathbb{R}^n$ ,  $f = \{f_s : U_s \rightarrow \mathbb{R}\}_{s \in \mathbb{R}^m}$  a definable family of functions,  $k \in \mathbb{N}$ . Then there is  $V \subset U$  a definable family of open subsets such that for each  $s \in \mathbb{R}^m$ ,  $U_s \setminus V_s$  is nowhere dense and the restriction  $f_s|_{V_s}$  is  $C^k$ -smooth.*

*Proof.* For  $s \in \mathbb{R}^m$ , set

$$V_s := \{x \in U_s \mid f_s \text{ is } C^k\text{-smooth at } x\}.$$

Then  $V = \{V_s\}_{s \in \mathbb{R}^m}$  is definable and  $V_s$  is open. Moreover, by definition of  $V_s$ ,  $(f_s)|_{V_s}$  is  $C^k$ -smooth. Left to show is that  $U_s \setminus V_s$  is nowhere dense. But this follows from d-minimal generic smoothness applied to  $f_s$ .  $\square$

**Theorem 2.3.9** (Family Version of Countable Cell Decomposition). *Work in a d-minimal structure expanding the real field. For any  $k \in \mathbb{N}$  and  $A = \{A_s\}_{s \in \mathbb{R}^m}$  a definable family of subsets of some  $\mathbb{R}^n$ , there are countably many families of cells  $C^\alpha = \{C_s^\alpha\}_{s \in \mathbb{R}^m}$  ( $\alpha \in \Delta$  for some countable index set  $\Delta$ ) s.t. for a fixed  $s \in \mathbb{R}^m$ , the collection  $\{C_s^\alpha \mid \alpha \in \Delta\}$  forms a countable  $C^k$ -decomposition of  $\mathbb{R}^n$  compatible with  $A_s$  (here cells can be empty).*

*Proof.* Apply countable cell decomposition to  $A$ , this gives a countable  $C^k$ -decomposition of  $\mathbb{R}^m \times \mathbb{R}^n$  compatible with  $A$ , say  $\{\mathcal{C}^\alpha \mid \alpha \in \Delta\}$  for some countable index set  $\Delta$ . Now each  $\mathcal{C}^\alpha \subset \mathbb{R}^m \times \mathbb{R}^n$  can itself be regarded as a definable family  $\{\mathcal{C}_s^\alpha\}_{s \in \mathbb{R}^m}$ . For fixed  $s \in \mathbb{R}^m$ , the fiber  $\mathcal{C}_s^\alpha \subset \mathbb{R}^n$  is again a cell, possibly empty, and the collection  $\{\mathcal{C}_s^\alpha \mid \alpha \in \Delta\}$  forms a countable  $C^k$ -decomposition of  $\mathbb{R}^n$  (cf. [13, Proposition 3.3.5]). It is compatible with  $A_s$  as the original decomposition  $\{\mathcal{C}^\alpha \mid \alpha \in \Delta\}$  is compatible with  $A$ .  $\square$

Finally, also in d-minimal structures we have definable choice, an important tool in many proofs:

**Theorem 2.3.10** (Definable Choice, Miller, [27]). *Let  $\mathcal{R}$  be a d-minimal expansion of a dense ordered group.*

- (Definable Skolem Functions) *If  $A \subset R^{m+n}$  is definable and  $\pi : R^{m+n} \rightarrow R^m$  is the projection on the first  $m$  coordinates, then there is a definable  $f : \pi A \rightarrow R^n$  such that  $\Gamma(f)$  is contained in  $A$ .*
- (Elimination of Imaginaries) *For any definable equivalence relation on a definable set  $A$  there exists a definable set of representatives.*

Of course, the full strength of o-minimal geometry cannot be reached within the d-minimal setting. A major drawback, for example, is the lack of uniform finiteness: By a lemma of Miller, any d-minimal expansion of  $(\mathbb{R}, <, +)$  that has the uniform finiteness property is already o-minimal [26, §5]. The aim of Section 2.6 will be to establish a weaker version of uniform finiteness for certain d-minimal structures.

## 2.4 Dimension

In the o-minimal setting there is a well-behaved notion of dimension for definable sets  $X$ :

$$\dim X = \max\{i_1 + \dots + i_n \mid X \text{ contains an } (i_1, \dots, i_n)\text{-cell}\}$$

We can generalize this to arbitrary subsets of  $\mathbb{R}^n$ :

**Definition 2.4.1.** Let  $X \subset \mathbb{R}^n$  for some  $n \in \mathbb{N}$ . The **dimension** of  $X$ , denoted by  $\dim X$ , is the supremum of all  $k \in \mathbb{N}$ ,  $k \leq n$  such that for some coordinate projection  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,  $\pi(X)$  has interior.  $\dim \emptyset$  is set to  $-\infty$ .

For definable sets in o-minimal structures over the reals this agrees with the definition above. In this section we want to further examine this dimension in tame expansions of the real field and compare it with other, metric notions of dimension. Again, most proofs are omitted. We start by showing that d-minimality already implies that definable 0-dimensional subsets of any  $\mathbb{R}^n$  (not only subsets of  $\mathbb{R}$ ) are finite union of discrete sets - also uniformly in definable families.

**Proposition 2.4.2.** *An expansion  $\mathcal{R}$  of  $(\mathbb{R}, <)$  is d-minimal, iff for every  $m, n \in \mathbb{N}$  and definable  $A \subset \mathbb{R}^m \times \mathbb{R}^n$  there exists  $N \in \mathbb{N}$  such that for every  $x \in \mathbb{R}^m$ , either  $\dim A_x > 0$  or  $A_x$  is the union of at most  $N$  discrete sets.*

*Proof.* We take Condition (2) from Proposition 2.1.7 as the definition of d-minimality. " $\Leftarrow$ " is clear. For the other direction, proceed by induction on  $n$ , the base case is taken care of by condition (2) in Proposition 2.1.7. Assume the claim holds for  $n$  and let  $A \subset \mathbb{R}^m \times \mathbb{R}^{n+1}$  be given. Apply the assumption to all projections of  $A$  onto  $\mathbb{R}^m \times \mathbb{R}^n$  and set  $N'$  to be

the maximum of the resulting  $N$ 's. Then for  $x \in \mathbb{R}^m$  with  $\dim A_x = 0$ , we know that all projections of  $A_x \subset \mathbb{R}^{n+1}$  to  $\mathbb{R}^n$  are the union of at most  $N'$  discrete sets. Thus  $A_x$  is the union of at most  $N = (N')^n$  sets whose projections to  $\mathbb{R}^n$  are all discrete. Then also these  $N$  sets must be discrete.  $\square$

The dimension has the following tame properties in noiseless structures:

**Proposition 2.4.3** (Miller, [26, §7, Main Lemma]). *Let  $\mathcal{R}$  be a structure expanding the real field. The following are equivalent:*

- (1) *The structure  $\mathcal{R}$  is noiseless*
- (2)  *$\dim(X) = \dim(\text{cl}(X))$  for all definable sets  $X$*
- (3)  *$\dim \bigcup_{k \in \mathbb{N}} X_k = \max_{k \in \mathbb{N}} \{\dim X_k\}$  for any definable subsets  $X_1, X_2, \dots$  of  $\mathbb{R}^n$*

**Remark 2.4.4.** Not all the nice properties from o-minimality hold in this general setting: For example, in d-minimal structures we do not have  $\dim \text{fr } X < \dim X$  for any definable set  $X$ . A counterexample is given by  $X = 2^{\mathbb{Z}}$ , then  $\text{fr } X = \{0\}$  and  $\dim \text{fr } X = 0 = \dim X$ .

Now we turn towards two different notions of metric dimension. It turns out that they both agree with the usual dimension in d-minimal structures. Fix some  $\mathbb{R}^n$  with its usual metric, denoted by  $d$ .

**Definition 2.4.5.** For  $X \subset \mathbb{R}^n$  and  $r > 0$  put

$$\text{net}_r X = \sup\{k \in \mathbb{N} : \exists x_1, \dots, x_k \in X, \bigwedge_{i \neq j} d(x_i, x_j) \geq r\}$$

**Remark 2.4.6.**  $\text{net}_r X$  is closely related to the number of balls of radius  $r$  needed to cover  $X$ : In fact, one easily sees that if  $\text{net}_r X$  is finite, one can always cover  $X$  with  $\text{net}_r X$  balls of radius  $r$ . On the other hand, one needs at least  $\frac{1}{N} \cdot \text{net}_r X$  balls of radius  $r$  to cover  $X$ , where  $N \in \mathbb{N}$  only depends on  $n$ : Take  $x_1, \dots, x_{\text{net}_r X}$  with  $d(x_i, x_j) \geq r$  for  $i \neq j$ . Then any ball with radius  $r$  can cover at most  $N$  of these for some  $N$  dependent on  $n$  (e.g. for  $\mathbb{R}$  this works with  $N = 2$ ).

**Remark 2.4.7.** We have  $\text{net}_r X = \infty$ , iff  $X$  is unbounded:

If  $X$  is unbounded, we can always find arbitrarily many elements with pairwise distance greater than a given  $r$ . If  $X$  is bounded, it can be covered by finitely many balls of given radius  $r$ , which by the previous remark implies that  $\text{net}_r X$  is finite.

**Definition 2.4.8.** Let  $X \subset \mathbb{R}^n$  be bounded and nonempty. The **upper Minkowski dimension** (also called **upper box dimension**) of  $X$  is given by

$$\overline{\dim}_M X = \overline{\lim}_{r \rightarrow 0} \frac{\log \text{net}_r X}{-\log r},$$

that is,  $\overline{\dim}_M X$  is the supremum of all  $\alpha \in \mathbb{R}$  for which there is a sequence  $(r_i)_{i \in \mathbb{N}}$  of positive real numbers converging to 0 with

$$\lim_{i \rightarrow \infty} \frac{\log \text{net}_{r_i} X}{-\log r_i} = \alpha.$$

Now let  $X \subset \mathbb{R}^n$  be arbitrary. The **Assouad dimension** of  $X$ , denoted  $\text{Dim } X$ , is the infimum of  $\alpha \in \mathbb{R}$  such that

$$\{(r/R)^\alpha \text{net}_r \{y \in X : d(y, x) \leq R\} : x \in X, 0 < r < R < \infty\}$$

is bounded.

To give some intuition: The upper Minkowski dimension measures the growth rate of  $\text{net}_r X$  for  $r \rightarrow 0$ . The Assouad dimension adds a level of complexity and measures the growth rate of  $\text{net}_r(X \cap B_R)$  for  $r \rightarrow 0$  and  $R \rightarrow \infty$  where  $B_R$  is a ball with radius  $R$  around some element of  $X$ . The general relationship between  $\dim$ ,  $\overline{\dim}_M$  and  $\text{Dim}$  is the following:

**Proposition 2.4.9.** *For  $X \subset \mathbb{R}^n$ ,  $X$  nonempty bounded, we always have*

$$\dim X \leq \overline{\dim}_M X \leq \text{Dim } X.$$

Hieronymi and Miller prove that in noiseless structures all these notions agree:

**Theorem 2.4.10** (Hieronymi and Miller, [24, Corollary 1.4]). *Let  $\mathcal{R}$  be a structure expanding the real field. The following are equivalent:*

- (1) *The structure  $\mathcal{R}$  is noiseless.*
- (2)  *$\dim X = \text{Dim } X$  for all definable sets  $X$ .*

*In particular, by Proposition 2.4.9, in this case also  $\overline{\dim}_M X = \dim X$  for nonempty bounded  $X$ .*

**Remark 2.4.11.** A particularly nice theory of dimension can be developed for so-called **strongly noiseless** structures, these are noiseless structures in which no Cantor subset of  $\mathbb{R}$  is definable (i.e. a nonempty, compact, totally disconnected set without isolated points). Note that any d-minimal structure is strongly noiseless.

Hieronymi and Miller prove that in strongly noiseless structures, on definable sets  $\dim$  moreover agrees with the so-called small inductive dimension, one of the most usual topological dimensions. This amounts for saying that in this case, "all dimensions commonly encountered in geometric measure theory, fractal geometry and analysis on metric spaces are equal" [24].

## 2.5 Metric properties of 0-dimensional definable subsets of $\mathbb{R}$

In this section we want to apply Theorem 2.4.10 from last section to analyze metric properties of 0-dimensional subsets of  $\mathbb{R}$  definable in noiseless structures.

The underlying idea is that the points in such a set cannot lie too evenly distributed over the real line - otherwise, the integers would be definable.

In a second step we take a look at the d-minimal structure  $(\mathcal{R}, \alpha^{\mathbb{Z}})$  for some o-minimal expansion of the real field  $\mathcal{R}$  with field of exponents  $\mathbb{Q}$ ,  $\alpha \in \mathbb{R}$ ,  $\alpha > 1$ . Based on the work of Tychonievich, we can give a more accurate description of 0-dimensional subsets of  $\mathbb{R}$  definable in this structure, focusing on the convergence rate of definable sequences.

### 2.5.1 In noiseless structures

For this subsection fix a noiseless expansion of the real field. Again, we equip  $\mathbb{R}^n$  with the usual metric.

Unwinding the definition of the upper Minkowski dimension and using that it agrees with the usual dimension we get the following: In noiseless structures,  $\text{net}_r X$  does grow rather slow with  $r \rightarrow 0$  for definable 0-dimensional sets. This formalizes the fact that points tend to accumulate rather than being evenly distributed.



**Proposition 2.5.1.** *Let  $X \subset \mathbb{R}^n$  be definable, bounded of (usual) dimension 0,  $\varepsilon \in (0, 1)$ . Then there exists a constant  $C \in \mathbb{R}$  s.t. for all  $r > 0$  we have*

$$\text{net}_r X \leq C \cdot r^{-\varepsilon}$$

*Proof.* By Theorem 2.4.10,  $\overline{\dim}_M X = 0$ . By definition of the upper Minkowski dimension, this implies that there is  $r_0 \in (0, 1)$  s.t. for all  $r \in (0, r_0)$  we have:

$$\frac{\log \text{net}_r X}{-\log r} \leq \varepsilon$$

This readily implies (using  $r_0 < 1$ )

$$\text{net}_r X \leq r^{-\varepsilon} \quad \text{for } r \in (0, r_0).$$

Moreover, as  $X \subset \mathbb{R}^n$  is bounded, it can be covered by  $C \cdot r^{-n}$  balls of radius  $r$  for some constant  $C \in \mathbb{R}$  independent of  $r$ . By Remark 2.4.6, we get  $C \cdot r^{-n} \geq \frac{1}{N} \cdot \text{net}_r X$  for some  $N \in \mathbb{N}$  independent of  $r$ , hence for all  $r > 0$

$$\text{net}_r X \leq C' \cdot r^{-n}$$

for some constant  $C' \in \mathbb{R}$  independent of  $r$ . In particular for  $r \geq r_0$  (using  $\varepsilon < 1 \leq n$ ):

$$\text{net}_r X \leq C' \cdot r^{\varepsilon-n} \cdot r^{-\varepsilon} \leq C' \cdot r_0^{\varepsilon-n} \cdot r^{-\varepsilon}$$

Putting  $C = \max\{1, C' \cdot r_0^{\varepsilon-n}\}$  we get

$$\text{net}_r X \leq C \cdot r^{-\varepsilon}$$

for all  $r > 0$ . □

As a corollary, we may deduce that we have good control over the size of the set of points in a certain distance to a given definable 0-dimensional subset of  $\mathbb{R}$ :

**Corollary 2.5.2.** *Let  $X \subset \mathbb{R}$  be definable, bounded of dimension 0,  $\varepsilon \in (0, 1)$ . Then there exists a constant  $C \in \mathbb{R}$  s.t. for all  $r > 0$  we have*

$$\lambda\left(\bigcup_{x \in X} (x - r, x + r)\right) \leq C \cdot r^\varepsilon,$$

where  $\lambda$  denotes the Lebesgue measure.

*Proof.* By Remark 2.4.6, we can cover  $X$  with  $\text{net}_r X$  intervals of length  $2r$ . If we enlarge each of these intervals at both sides by  $r$ , we get  $\text{net}_r X$  intervals of length  $4r$  that cover  $\bigcup_{x \in X} (x - r, x + r)$ . Thus, for all  $r > 0$ :

$$\lambda\left(\bigcup_{x \in X} (x - r, x + r)\right) \leq 4 \cdot r \cdot \text{net}_r X$$

The claim now follows immediately from Proposition 2.5.1, applied with  $1 - \varepsilon$  instead of  $\varepsilon$ . □

Finally, we also get control about the convergence rate of decreasing definable sequences with decreasing distance between neighbouring members of the sequence: These sequences must converge faster than any rational function.



**Proposition 2.5.3.** *Let  $(a_i)_{i \in \mathbb{N}}$  be a strictly decreasing sequence of positive real numbers converging to 0 s.t.  $A = \{a_i\}_{i \in \mathbb{N}}$  is definable and  $(d_i)_{i \in \mathbb{N}}$  is decreasing where  $d_i := a_i - a_{i+1}$ . Let  $t \in \mathbb{R}_+$ . Then there exists a constant  $C \in \mathbb{R}$  s.t. for all  $i \in \mathbb{N}$ :*

$$a_i \leq C \cdot i^{-t}$$

*Proof.* As  $(d_i)_{i \in \mathbb{N}}$  is decreasing, we know that for each  $n \in \mathbb{N}_+$ ,  $a_1, \dots, a_n$  are elements in  $A$  with pairwise distance at least  $d_{n-1}$ . By definition, this implies  $n \leq \text{net}_{d_{n-1}} A$ . On the other hand, Proposition 2.5.1 (applied with  $\varepsilon = \frac{1}{t+1}$ ) implies that there is a constant  $C' \in \mathbb{R}$  s.t. for all  $n \in \mathbb{N}_+$  we have  $\text{net}_{d_{n-1}} A \leq C' \cdot d_{n-1}^{-\varepsilon}$ . Combining this, we get for each  $n \in \mathbb{N}_+$ :

$$\begin{aligned} n &\leq C' \cdot d_{n-1}^{-\frac{1}{t+1}} \\ \Leftrightarrow d_{n-1} &\leq (C')^{t+1} \cdot n^{-(t+1)} \end{aligned}$$

Now

$$\begin{aligned} a_n &= \sum_{i=n}^{\infty} d_i \\ &\leq C' \cdot \sum_{i=n}^{\infty} (i+1)^{-(t+1)} \\ &\leq C' \cdot \left( \frac{n^{1-(t+1)}}{t+1-1} + n^{-(t+1)} \right) \\ &\leq C \cdot n^{-t} \end{aligned}$$

for a suitable  $C \in \mathbb{R}$ , depending on  $t$  but not on  $n$ . In the second inequality we used a bound for partial sums of the Riemann zeta function that can be derived from one of the many definitions of the Riemann zeta function (cf. [7, Theorem 6.24]).  $\square$

**Remark 2.5.4.** Unfortunately, one cannot simply drop the condition that the  $d_i$  are decreasing - at least when using the approach via the upper Minkowski dimension: There are strictly decreasing sequences  $(a_i)_{i \in \mathbb{N}}$  of positive real numbers converging to 0 with  $\overline{\dim}_M \{a_i \mid i \in \mathbb{N}\} = 0$ , such that the conclusion of the proposition does not hold. However, it remains open whether such a sequence can be definable in a noiseless structure.

## 2.5.2 In $(\mathcal{R}, \alpha^{\mathbb{Z}})$

Fix  $\mathcal{R}$ , an o-minimal expansion of the real field with field of exponents  $\mathbb{Q}$ , and  $\alpha \in \mathbb{R}$  with  $\alpha > 1$ . In this subsection we work in the structure  $(\mathcal{R}, \alpha^{\mathbb{Z}})$  which we have seen to be d-minimal in Section 2.2. Denote  $H := \alpha^{-\mathbb{N}_+} \subset I$  to simplify notation.

Tychonievich presented a powerful characterization of the definable sets in this structure: They are finite unions of images of  $H^m \times I^d$  (for some  $m, d \in \mathbb{N}$ ) under sufficiently well-behaved functions that are already definable in  $\mathcal{R}$  [40]. While Tychonievich aims to prove a metric criterion when  $\dim \text{fr } X < \dim X$  for definable sets  $X$ , we will use his methods here to improve the results from last subsection on 0-dimensional subsets of  $\mathbb{R}$ . In particular, we will show that any convergent definable sequence  $(a_i)_{i \in \mathbb{N}}$  converges faster than  $e^{-i^B}$  for some  $B \in \mathbb{R}_+$ :

**Theorem 2.5.5.** *Let  $(a_i)_{i \in \mathbb{N}}$  be a strictly decreasing sequence of positive real numbers converging to 0, s.t.  $\{a_i \mid i \in \mathbb{N}\}$  is definable. Then there exist  $A, B, C \in \mathbb{R}_+$  s.t. for all  $i \in \mathbb{N}$ :*

$$a_i \leq C \cdot e^{-Ai^B}$$

From this we can deduce a more general statement regarding the structure of arbitrary 0-dimensional definable subsets of  $\mathbb{R}$ :

**Theorem 2.5.6.** *Let  $Z \subset I$  be definable of dimension 0.*

(i) *There are  $a, b, c \in \mathbb{R}_+$  s.t. for all  $r \in \mathbb{R}_+$ :*

$$\#(Z \setminus \bigcup_{z \in \lim(Z)} (z - r, z + r)) \leq (-a \cdot \log(c \cdot r))^b$$

(ii) *There are  $a', b', c' \in \mathbb{R}_+$  s.t. for all  $r \in \mathbb{R}_+$ :*

$$\text{net}_r Z \leq (-a' \cdot \log(c' \cdot r))^{b'}$$

**Remark 2.5.7.** Note that we cannot really improve the bound in Theorem 2.5.5. In particular, the exponent  $B$  may not be dropped: Consider the definable set  $\{x + \frac{1}{4}xy \mid x, y \in 2^{-\mathbb{N}}, x < y\}$ . The corresponding sequence converges only as fast as  $e^{-\sqrt{i}}$ . Similar examples will give convergence rate  $e^{-i^B}$  for arbitrarily small  $B \in \mathbb{R}_+$

First note that in order to prove Theorem 2.5.5 for a finite union of sets, it suffices to prove it for each set individually:

**Lemma 2.5.8.** *Let  $Z_1, \dots, Z_n$  be subsets of  $I$  with unique limit point 0 s.t. the conclusion of Theorem 2.5.5 holds for each  $Z_k$ , i.e. for each  $k \in \{1, \dots, n\}$  there exist  $A_k, B_k, C_k \in \mathbb{R}_+$  s.t. if  $(a_i^k)_{i \in \mathbb{N}}$  is a strictly decreasing enumeration of  $Z_k$ , then for each  $i \in \mathbb{N}$ :*

$$a_i^k \leq C_k \cdot e^{-A_k i^{B_k}}$$

*Then the conclusion of Theorem 2.5.5 also holds for  $Z := \bigcup_{j=1}^n Z_j$ , i.e. there are  $A, B, C \in \mathbb{R}_+$  s.t. if  $(a_i)_{i \in \mathbb{N}}$  is a strictly decreasing enumeration of  $Z$ , then for each  $i \in \mathbb{N}$ :*

$$a_i \leq C \cdot e^{-A i^B}$$

*Proof.* For each  $i \in \mathbb{N}$  there are  $k \in \{1, \dots, n\}$  and  $j \in \mathbb{N}$ , with  $a_i = a_j^k$ . This implies, by the pigeonhole principle and since  $(a_i)_{i \in \mathbb{N}}$  is decreasing, that for each  $i \in \mathbb{N}$  there exists  $k \in \{1, \dots, n\}$  and  $j \in \mathbb{N}$  with  $j \geq \frac{i}{n}$  s.t.  $a_i \leq a_j^k$ . But then, by assumption,

$$a_i \leq a_j^k \leq C_k \cdot e^{-A_k j^{B_k}} \leq C_k \cdot e^{-A_k (\frac{i}{n})^{B_k}}$$

This shows the claim with  $C := \max\{C_k \mid k = 1, \dots, n\}$ ,  $B := \min\{B_k \mid k = 1, \dots, n\}$ ,  $A := \min\{A_k \mid k = 1, \dots, n\} \cdot (\frac{1}{n})^B$ .  $\square$

We continue the proof of Theorem 2.5.5 with a reduction step exploiting Tychonievich's classification of definable sets in the structure  $(\mathcal{R}, \alpha^{\mathbb{Z}})$ .

**Reduction Step.** *Reduction to the case  $\{a_i \mid i \in \mathbb{N}\} = f(H^m) \setminus \{0\}$  for some  $m \in \mathbb{N}$  and some function  $f : I^m \rightarrow [0, 1)$ , definable in  $\mathcal{R}$ , with*

$$\lim_{z \rightarrow h, z \in H^m} f(z) = 0 \quad \text{for all } h \in \text{fr } H^m.$$

*Proof.* Let  $Z := \{a_i \mid i \in \mathbb{N}\}$ . As finitely many members of the sequence do not matter for the conclusion of the theorem, we may assume  $Z \subset I$ . By [40, Theorem 4.2.1] there are finitely many cells  $\mathcal{C}_k \subset I^{m_k}$  and continuous functions  $\bar{f}_k : \mathcal{C}_k \times I^{d_k} \rightarrow \mathbb{R}$  (with  $m_k, d_k \in \mathbb{N}$ ) such that  $Z$  is the disjoint union of the images  $\bar{f}_k((\mathcal{C}_k \cap H^{m_k}) \times I^{d_k})$  and such that for each  $k$ :

- (i)  $\bar{f}_k$  is definable in  $\mathcal{R}$ .
- (ii) For each  $x \in \mathcal{C}_k$ , the function  $y \mapsto \bar{f}_k(x, y)$  is injective.
- (iii)  $\bar{f}_k$  is injective on the set  $(\mathcal{C}_k \cap H^{m_k}) \times I^{d_k}$ .

As  $Z$  is countable, the second property implies  $d_k = 0$  for all  $k$ . Moreover, by Lemma 2.5.8, we can prove our claim on each of the sets  $\bar{f}_k((\mathcal{C}_k \cap H^{m_k}) \times I^{d_k})$  individually. (Here we can ignore the finite sets of this form as forgetting finitely many elements doesn't matter for the claim to prove. The infinite sets must have 0 as unique limit point and we can apply the lemma). Hence wlog we may assume that there is only one cell  $\mathcal{C} \subset I^m$  for some  $m \in \mathbb{N}$  and a continuous function  $\bar{f} : \mathcal{C} \rightarrow \mathbb{R}$  definable in  $\mathcal{R}$  s.t.  $Z = \bar{f}(\mathcal{C} \cap H^m)$  and  $\bar{f}$  is injective on  $\mathcal{C} \cap H^m$ .

Define  $f : I^m \rightarrow [0, 1)$  as follows:

$$f(x) = \begin{cases} \bar{f}(x) & \text{for } x \in \mathcal{C} \text{ and } \bar{f}(x) \in I \\ 0 & \text{otherwise} \end{cases}$$

$f$  is definable in  $\mathcal{R}$  and we have  $Z = f(H^m) \setminus \{0\}$  and  $f$  is injective on  $H^m \cap \text{supp}(f)$ , where  $\text{supp}(f) = \{x \in I^m \mid f(x) \neq 0\}$ , the support of  $f$ .

**Claim.** For all  $h \in \text{fr } H^m$  we have  $\lim_{z \rightarrow h, z \in H^m} f(z) = 0$ .

*Proof of Claim.* Let  $h \in \text{fr } H^m$ ,  $(z_i)_{i \in \mathbb{N}}$  a sequence in  $H^m$  converging to  $h$ . Assume for a contradiction that  $\limsup_{i \rightarrow \infty} f(z_i) > 0$ . Then the sequence  $(\sup\{f(z_j) \mid j \geq i\})_{i \in \mathbb{N}}$  cannot be eventually constant as  $(z_i)_{i \in \mathbb{N}}$  is not and  $f$  is injective on  $H^m \cap \text{supp}(f)$ . Thus,  $\limsup_{i \rightarrow \infty} f(z_i)$  must be a limit point of  $\{f(z_i) \mid i \in \mathbb{N}\} \subset Z \cup \{0\}$ . But 0 is the unique limit point of  $Z$ .  $\square$

This finishes the proof of the reduction step.  $\square$

We now follow the lines of the proof of [40, Theorem 4.4.2]. We introduce some notation: For  $h = (h_1, \dots, h_m) \in H^m$ , set  $|h| := \min\{h_1, \dots, h_m\}$ . Moreover, define  $\Lambda : \mathbb{R} \rightarrow H \cup \{0\}$  by

$$\Lambda(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \alpha^n & \text{for } n \in \mathbb{Z} \text{ s.t. } \alpha^n \leq x < \alpha^{n+1} \end{cases}$$

Note that the structures  $(\mathcal{R}, H)$  and  $(\mathcal{R}, \Lambda)$  are interdefinable. We are ready to prove the theorem now:

*Proof of Theorem 2.5.5.* By our reduction step, we may wlog assume  $Z := \{a_i \mid i \in \mathbb{N}\} = f(H^m) \setminus \{0\}$  for some  $m \in \mathbb{N}$  and some function  $f : I^m \rightarrow [0, 1)$ , definable in  $\mathcal{R}$ , with  $\lim_{z \rightarrow h, z \in H^m} f(z) = 0$  for all  $h \in \text{fr } H^m$ .

Define  $g : H \rightarrow H$  as follows:

$$g(x) = \max\{(\Lambda \circ f)(h) \mid h \in H^m \cap \text{supp}(f), |h| = x\}$$

This is well-defined as the set  $\{h \in H^m \mid |h| = x\}$  is finite for any given  $x \in H$ . The proof proceeds by closer analyzing the function  $g$ . We start with the following key properties:

**Claim.** (i)  $g$  is definable in the structure  $(H, \cdot, <)$ .

(ii)  $\lim_{x \in H, x \rightarrow 0} g(x) = 0$

*Proof of Claim.* (i): Certainly,  $(\Lambda \circ f)(\text{supp}(f)) \subset H$ , hence the graph of  $\Lambda \circ f : H^m \cap \text{supp}(f) \rightarrow H$  is a subset of  $H^{m+1}$  definable in  $(\mathcal{R}, H)$ . By [40, Theorem 4.1.2], this implies, that  $\Lambda \circ f : H^m \cap \text{supp}(f) \rightarrow H$  is even definable in  $(H, \cdot, <)$ . The same holds for  $\max$ ,  $\min$  and  $|\cdot|$ , hence also for  $g$ .

(ii): Assume for a contrary there was an  $\varepsilon \in \mathbb{R}_+$  s.t. there are arbitrarily small  $x \in H$  with  $g(x) \geq \varepsilon$ . By definition of  $g$  this implies, that there is an infinite subset  $M$  of  $H^m$  s.t.  $(\Lambda \circ f)|_M \geq \varepsilon$ . But then  $M$  must have a limit point  $h$  which then must be part of  $\text{fr}(H^m)$  and there would be a sequence  $(z_i)_{i \in \mathbb{N}}$  of elements of  $H^m$  converging to  $h$  with  $f(z_i) \geq \varepsilon$  for all  $i$ . This is a contradiction to  $\lim_{z \rightarrow h, z \in H^m} f(z) = 0$ .  $\square$

Now, the structure  $(H, \cdot, <)$  is isomorphic to the structure  $(\mathbb{N}, +, <)$ . Using a result of Cluckers [6, Theorem 1] we can deduce that there exists a finite partition of  $H$  s.t. on each component  $C$  we have  $g(\alpha^k) = \alpha^{\beta+\gamma k}$  for some  $\beta, \gamma \in \mathbb{Q}$ . We can ignore the finite components (finitely many elements of our sequence do not matter in the end), for the infinite components we must have  $\gamma > 0$  as  $\lim_{x \in H, x \rightarrow 0} g(x) = 0$ . Thus, taking the maximal such  $\beta$  and the minimal such  $\gamma$ , we can find  $\beta \in \mathbb{Q}, \gamma \in \mathbb{Q}_+$  s.t.:

$$g(\alpha^k) \leq \alpha^{\beta+\gamma k} \quad \text{for all } k \in \mathbb{Z}_{<0}. \quad (2.1)$$

Now enumerate  $H^m$  according to descending  $|h|$ :  $h_1$  is the unique  $h \in H^m$  with  $|h| = \alpha^{-1}$ ,  $h_2, \dots, h_{2^m}$  are the  $h \in H^m$  with  $|h| = \alpha^{-2}$ , and so on. Then one easily checks that

$$|h_i| \leq \alpha^{-i \frac{1}{m}} \quad \text{for all } i \in \mathbb{N}_+. \quad (2.2)$$

Define a sequence  $(b_i)_{i \in \mathbb{N}_+}$  by  $b_i := f(h_i)$ . As  $Z = f(H^m \cap \text{supp}(f))$ , we know that  $(b_i)_{i \in \mathbb{N}_+}$  enumerates  $Z$  (and potentially some of the  $b_i = 0$ ). Putting everything so far together, we get for all  $i \in \mathbb{N}_+$ :

$$\begin{aligned} b_i &= f(h_i) \\ &\leq \alpha \cdot \Lambda(f(h_i)) && \text{by definition of } \Lambda \\ &\leq \alpha \cdot g(|h_i|) && \text{by definition of } g \\ &\leq \alpha^{\beta+1+\gamma \cdot (-i \frac{1}{m})} && \text{by (2.1)+(2.2)} \end{aligned} \quad (2.3)$$

The bound on the  $b_i$  implies the same bound on the  $a_i$  as follows: For all  $i \in \mathbb{N}_+$  we know that there are  $i$  elements of  $Z$  greater or equal than  $a_i$ , as  $(a_i)_{i \in \mathbb{N}_+}$  is strictly decreasing. As also  $(b_i)_{i \in \mathbb{N}_+}$  enumerates  $Z$ , by the pigeonhole principle there must be  $j \geq i$  with  $b_j \geq a_i$ . This implies

$$a_i \leq b_j \stackrel{(2.3)}{\leq} \alpha^{\beta+1+\gamma \cdot (-j \frac{1}{m})} \leq \alpha^{\beta+1+\gamma \cdot (-i \frac{1}{m})},$$

yielding the claim.  $\square$

An equivalent reformulation of Theorem 2.5.5 is the following:

**Corollary 2.5.9.** *Let  $Z \subset I$  be definable of dimension 0 with unique limit point 0. Then there are  $a, b, c \in \mathbb{R}_+$  s.t. for all  $r \in \mathbb{R}_+$ :*

$$\#(Z \setminus [0, r)) \leq (-a \cdot \log(c \cdot r))^b$$

*Proof.* Let  $(a_i)_{i \in \mathbb{N}_+}$  be a strictly decreasing sequence in  $I$  converging to 0 enumerating  $Z$ . By Theorem 2.5.5 there exist  $A, B, C \in \mathbb{R}_+$  s.t. for all  $i \in \mathbb{N}$ :

$$a_i \leq C \cdot e^{-Ai^B}$$

For  $r \in \mathbb{R}_+$  this yields

$$\begin{aligned} \#Z \setminus [0, r) &= \max\{i \mid a_i \geq r\} \\ &\leq \max\{i \mid C \cdot e^{-Ai^B} \geq r\} \\ &= \max\{i \mid i \leq (-A^{-1} \cdot \log(C^{-1} \cdot r))^{B^{-1}}\} \\ &\leq (-a \cdot \log(c \cdot r))^b \end{aligned}$$

with  $a = A^{-1}, b = B^{-1}, c = C^{-1}$ . □

Finally, we can generalize this to 0-dimensional sets with arbitrarily many limit points while simultaneously proving a similar bound on  $\text{net}_r Z$ :

*Proof of Theorem 2.5.6.* Replacing  $Z$  by its closure we can wlog assume  $Z$  closed (now a subset of  $[0, 1]$ ). Using induction over the Cantor-Bendixson hierarchy, we may further assume that the claims hold for  $\lim(Z)$  (here we use that  $Z$  has finite Cantor-Bendixson rank by d-minimality).

If  $\lim(Z) = \emptyset$ ,  $Z$  must be finite, as it is bounded. Then the claims clearly hold. So assume  $\lim(Z)$  nonempty.

As  $\lim(Z)$  is definable, closed and nonempty, the function  $\delta := d(-, \lim(Z)) : Z \rightarrow [0, 1]$ , the distance to  $\lim(Z)$  in the usual metric on  $\mathbb{R}$ , is definable. Now  $\delta(Z) \subset [0, 1]$  is clearly 0-dimensional, as  $Z$  is so. Moreover, for each  $\varepsilon \in \mathbb{R}_+$ ,  $\delta(Z) \cap (\varepsilon, 1]$  is finite: Otherwise,  $\{z \in Z \mid d(z, \lim(Z)) > \varepsilon\}$  was infinite, implying that there exists a limit point of  $Z$  with distance at least  $\varepsilon$  to  $\lim(Z)$  - a contradiction. This shows that 0 is the unique limit point of  $\delta(Z)$ . Applying Corollary 2.5.9, we get  $a_1, b_1, c_1 \in \mathbb{R}_+$  s.t. for all  $r \in \mathbb{R}_+$ :

$$\#(\delta(Z) \setminus [0, r)) \leq (-a_1 \cdot \log(c_1 \cdot r))^{b_1}. \quad (2.4)$$

If  $\delta$  was injective, this would directly imply the first claim. However, we need another bound on the cardinality of the fibers of  $\delta$ . Let  $r$  be a nonzero element of  $\delta(Z)$ , we want to bound  $\#\delta^{-1}(\{r\})$  (we already know that this is finite as otherwise  $Z$  would have a limit point with distance at least  $r$  to  $\lim(Z)$ ).

For  $z \in \delta^{-1}(\{r\})$ , let  $z' \in \lim(Z)$  be a nearest element in  $\lim(Z)$ , so  $d(z, z') = r$ . Consider the set of all these  $z'$ ,  $Z' := \{z' \mid z \in \#\delta^{-1}(\{r\})\} \subset \lim(Z)$  (finite). As for each point in  $Z'$  there exist only two points with distance precisely  $r$ , we see that  $\#\delta^{-1}(\{r\}) \leq 2 \cdot \#Z'$ . Moreover, taking three points from  $Z'$ , the outer two points must have distance at least  $2r$ : Otherwise, there could not exist a point  $z$  with distance  $r$  to the middle point, such that the middle point is the nearest point of  $\lim(Z)$  to  $z$ .

Thus, taking every second point of  $Z'$ , we see that  $\frac{1}{2} \cdot \#Z' \leq \text{net}_{2r} \lim(Z)$ . Together, we have shown

$$\begin{aligned} \#\delta^{-1}(\{r\}) &\leq 4 \cdot \text{net}_{2r} \lim(Z) \\ &\leq (-a_2 \cdot \log(c_2 \cdot r))^{b_2} \end{aligned} \quad (2.5)$$

for some constants  $a_2, b_2, c_2 \in \mathbb{R}_+$  independent of  $x$ , by assumption on  $\lim(Z)$ .

Now we combine everything to get, for all  $r \in \mathbb{R}_+$ ,

$$\begin{aligned}
\#(Z \setminus \bigcup_{z \in \lim(Z)} (z - r, z + r)) &= \#(\delta^{-1}([r, 1])) \\
&= \sum_{r' \in \delta(Z) \cap [r, 1]} \#\delta^{-1}(\{r'\}) \\
&\stackrel{(2.5)}{\leq} \sum_{r' \in \delta(Z) \setminus [0, r]} (-a_2 \cdot \log(c_2 \cdot r'))^{b_2} \\
&\leq \sum_{r' \in \delta(Z) \setminus [0, r]} (-a_2 \cdot \log(c_2 \cdot r))^{b_2} \\
&= \#(\delta(Z) \setminus [0, r]) \cdot (-a_2 \cdot \log(c_2 \cdot r))^{b_2} \\
&\stackrel{(2.4)}{\leq} (-a_1 \cdot \log(c_1 \cdot r))^{b_1} \cdot (-a_2 \cdot \log(c_2 \cdot r))^{b_2} \\
&\leq (-a \cdot \log(c \cdot r))^b
\end{aligned}$$

for suitable constants  $a, b, c \in \mathbb{R}_+$ . This shows the first claim.

In order to show the second, assume that  $z_1, \dots, z_n \in Z$  have pairwise distance at least  $r$  for some  $r \in \mathbb{R}_+$ . Let  $\bar{Z} := \{z_i \mid d(z_i, \lim(Z)) \leq \frac{1}{4} \cdot r\}$ . Then, by the first claim,

$$n \leq \#\bar{Z} + (-a \cdot \log(c \cdot \frac{1}{4}r))^b \quad (2.6)$$

Let  $Z' \subset \lim(Z)$  be the set of  $z' \in \lim(Z)$  s.t. there is  $z \in \bar{Z}$  with  $z'$  being the nearest element of  $\lim(Z)$ . In particular,  $d(z, z') \leq \frac{1}{4}r$  in this situation. As all elements of  $\bar{Z}$  have pairwise distance at least  $r$ , there cannot be two elements of  $\bar{Z}$  having the same nearest element in  $Z'$ . This implies  $\#\bar{Z} = \#Z'$ . Again, as all elements of  $\bar{Z}$  have pairwise distance at least  $r$ , all elements of  $Z'$  must have pairwise distance at least  $\frac{1}{2}r$ . This implies  $\#Z' \leq \text{net}_{\frac{1}{2}r} \lim(Z)$ . Using the assumption on  $\lim(Z)$ , we get constants  $a_3, b_3, c_3 \in \mathbb{R}_+$ , independent of  $r$  s.t.:

$$\begin{aligned}
&\stackrel{(2.6)}{n} \leq \#\bar{Z} + (-a \cdot \log(c \cdot \frac{1}{4}r))^b \\
&= \#Z' + (-a \cdot \log(c \cdot \frac{1}{4}r))^b \\
&\leq \text{net}_{\frac{1}{2}r} \lim(Z) + (-a \cdot \log(c \cdot \frac{1}{4}r))^b \\
&\leq (-a_3 \cdot \log(c_3 \cdot r))^{b_3} + (-a \cdot \log(c \cdot \frac{1}{4}r))^b \\
&\leq (-a' \cdot \log(c' \cdot r))^{b'}
\end{aligned}$$

for suitable constants  $a', b', c' \in \mathbb{R}_+$ , independent of  $r$ . As  $z_1, \dots, z_n$  were picked arbitrarily in  $Z$  with pairwise distance at least  $r$ , this implies the same bound for  $\text{net}_r Z$ , showing the second claim.  $\square$

## 2.6 A d-minimal version of uniform finiteness

Recall the uniform finiteness property from o-minimality:

**Theorem 2.6.1** (Uniform Finiteness). *Work in an o-minimal expansion of an ordered field  $R$ . Let  $A \subset R^m \times R$  be a definable family of finite subsets of  $R$ . Then there exists  $N \in \mathbb{N}$  s.t.  $\#A_x \leq N$  for all  $x \in R^m$ .*

This certainly does not hold in the d-minimal setting:

**Example 2.6.2.** Consider  $A = \{(x, y) \in I \times I \mid y \in 2^{\mathbb{Z}}, y > x\}$ , definable in the d-minimal structure  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ . Then  $A_x$  is finite for all  $x \in I$ , but  $\#A_x \rightarrow \infty$  for  $x \rightarrow 0$ .

Note that in this case  $\#A_x$  grows only as fast as  $\log(x^{-1})$  for  $x \rightarrow 0$ , so it grows slower than any rational function. The question arises if we have control about this growth rate in general. For a general definable family of finite subsets of  $I$ , say  $A \subset I \times I$ , we can describe the values of the parameter around which the cardinality of the fibers becomes unbounded, by  $L = \{x \in [0, 1] \mid \text{cl}(A)_x \text{ is infinite}\}$ . This leads us to the following naive conjecture, bounding the cardinality of fibers with parameters in a given distance of  $L$ :

**Conjecture.** *Let  $A \subset I \times I$  be a family of finite subsets of  $I$ , definable in a d-minimal structure on  $\mathbb{R}$ . Let  $L = \{x \in [0, 1] \mid \text{cl}(A)_x \text{ is infinite}\}$ . Let  $t \in \mathbb{R}_+$ . Then there exists a constant  $C \in \mathbb{R}$  s.t. for all  $r > 0$  and  $x \in I$  with  $d(x, L) \geq r$  we have:*

$$\#A_x \leq C \cdot r^{-t}$$

Unfortunately, again we can find a counterexample:

**Example 2.6.3.** Consider  $M = (2^{\mathbb{Z}} \cap I) \times (2^{\mathbb{Z}} \cap I)$ . Then all subsets of  $M$  are definable in the d-minimal structure  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$ .

Let  $A$  be a subset of  $M$  s.t.  $\#A_{2^k} = 2^{-2k}$  for all  $k \in \mathbb{Z}_{<0}$  (this is certainly possible as  $\#M_{2^k} = \infty$  for all  $k \in \mathbb{Z}_{<0}$ ). Then  $A$  satisfies the conditions of the above conjecture and  $L$  equals  $\{0\}$ .

Assume the conjecture holds (for  $t = 1$ ), then there exists  $C \in \mathbb{R}$  s.t. for all  $r \in \mathbb{R}_+$  and  $x \in I$  with  $x \geq r$  we have:  $\#A_x \leq C \cdot r^{-1}$ . In particular, for all  $k \in \mathbb{Z}_{<0}$ , setting  $r = x = 2^k$ , we get  $\#A_{2^k} \leq C \cdot 2^{-k}$ . As  $\#A_{2^k} = 2^{-2k}$ , this implies  $2^{-k} \leq C$  for all  $k \in \mathbb{Z}_{<0}$ , a contradiction.

Using the method from the example one sees that the cardinality of the fibers may grow arbitrarily fast. The underlying reason is that the structure  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$  allows for the existence of definable sequences that converge arbitrarily fast.

However, if we restrict ourselves to d-minimal structures where definable sequences may not converge arbitrarily fast, the picture changes. Consider the following assumption on a d-minimal structure:

**Assumption (\*)**. *Let  $(a_i)_{i \in \mathbb{N}}$  be a strictly decreasing sequence of positive real numbers converging to 0, s.t.  $\{a_i \mid i \in \mathbb{N}\}$  is definable. Then there exist  $A, B, C, A', B', C' \in \mathbb{R}_+$  s.t. for all  $i \in \mathbb{N}$ :*

$$a_i \leq C \cdot e^{-Ai^B} \quad \text{and} \quad a_i \geq C' \cdot e^{-A'i^{B'}}$$

In the last section, we have seen that the structure  $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$  satisfies the first part of the assumption (Theorem 2.5.5), i.e. definable sequences cannot converge too slow. Presumably, also the second condition holds in this structure, although this has not been proven yet.

For the rest of this section fix a d-minimal expansion  $\mathcal{R}$  of the real line satisfying (\*). We aim to prove the following special version of uniform finiteness for  $\mathcal{R}$ :

**Theorem 2.6.4.** *Let  $A \subset I \times I$  be a definable family of finite subsets of  $I$ . Then there exists a 0-dimensional set  $L \subset [0, 1]$  and constants  $a, b, c \in \mathbb{R}_+$  s.t. for all  $r > 0$  and  $x \in I \setminus \bigcup_{x' \in L} (x' - r, x' + r)$  we have:*

$$\#A_x \leq (-a \cdot \log(c \cdot r))^b$$

We first prove a version for definable families where the parameter space is countable with unique limit point 0.

**Proposition 2.6.5.** *Let  $A \subset I \times I$  be a definable family of finite subsets of  $I$  s.t.  $\pi_1(A)$  is countable with unique limit point 0. Then there exist constants  $a, b, c \in \mathbb{R}_+$  s.t. for all  $x \in I$  we have*

$$\#A_x \leq (-a \cdot \log(c \cdot x))^b$$

*Proof.* Let  $(x_i)_{i \in \mathbb{N}}$  be a strictly decreasing sequence in  $I$  converging to 0 s.t.  $\{x_i \mid i \in \mathbb{N}\} = \pi_1(A)$  (possible by assumptions on  $\pi_1(A)$ ). Define another sequence by setting (for  $i \in \mathbb{N}$ )

$$r_i := \min\left(\bigcup_{j \leq i} (A_{x_j} - A_{x_j}) \setminus \mathbb{R}_{\leq 0}\right).$$

$r_i$  is the minimal distance between two different elements in the same fiber  $A_{x_j}$  for some  $j \leq i$ . This is well-defined and always positive as the minimum is taken over a finite set of positive real numbers. Moreover, the sequence  $(r_i)_{i \in \mathbb{N}}$  is decreasing as the set of which the minimum is taken becomes bigger with increasing  $i$ . We distinguish two cases:

Case 1:  $r_i \not\rightarrow 0$ . Then there exists an  $\varepsilon \in \mathbb{R}_+$  s.t. for all  $x \in I$  and  $y_1 \neq y_2 \in A_x$  we have  $d(y_1, y_2) \geq \varepsilon$ . But this implies that for all  $x \in I$ ,  $\#A_x \leq \frac{1}{\varepsilon} + 1$  and we have even a uniform bound on the cardinality of the fibers which directly gives the claim.

Case 2:  $r_i \rightarrow 0$ . Then  $(r_i)_{i \in \mathbb{N}}$  is a decreasing sequence of positive real numbers converging to 0 s.t.  $\{r_i \mid i \in \mathbb{N}\}$  is definable. The second part of Assumption  $(*)$  is easily shown to hold also for decreasing sequences, not necessarily strictly decreasing. We get  $A_1, B_1, C_1 \in \mathbb{R}_+$  s.t.

$$r_i \geq C_1 \cdot e^{-A_1 i^{B_1}} \quad \text{for all } i \in \mathbb{N} \quad (2.7)$$

Also, by  $(*)$ , there are  $A_2, B_2, C_2 \in \mathbb{R}_+$  with

$$\begin{aligned} x_i &\leq C_2 \cdot e^{-A_2 i^{B_2}} && \text{for all } i \in \mathbb{N} \\ \Leftrightarrow i &\leq (-A_2^{-1} \cdot \log(C_2^{-1} \cdot x_i))^{B_2^{-1}} && \text{for all } i \in \mathbb{N} \end{aligned} \quad (2.8)$$

Note that, by definition of  $r_i$ , all elements in  $A_{x_i}$  have pairwise distance at least  $r_i$ . As all these lie in the same fiber, we have that also in  $\pi_2(A)$  there are at least  $\#A_{x_i}$  elements with pairwise distance at least  $r_i$  which implies

$$\#A_{x_i} \leq \text{net}_{r_i} \pi_2(A) \quad \text{for all } i \in \mathbb{N} \quad (2.9)$$

By Theorem 2.5.6 (which, by  $(*)$ , also holds in  $\mathcal{R}$  using the same proof as given above) there are  $a_1, b_1, c_1 \in \mathbb{R}_+$  s.t. for all  $r \in \mathbb{R}_+$ :

$$\text{net}_r \pi_2(A) \leq (-a_1 \cdot \log(c_1 \cdot r))^{b_1} \quad (2.10)$$



Putting everything together we get for all  $i \in \mathbb{N}$ :

$$\begin{aligned}
& \stackrel{(2.9)}{\#A_{x_i}} \leq \text{net}_{r_i} \pi_2(A) \\
& \stackrel{(2.10)}{\leq} (-a_1 \cdot \log(c_1 \cdot r_i))^{b_1} \\
& \stackrel{(2.7)}{\leq} (-a_1 \cdot \log(c_1 \cdot C_1 \cdot e^{-A_1 i^{B_1}}))^{b_1} \\
& = (-a_1 \cdot (\log(c_1 \cdot C_1) - A_1 i^{B_1}))^{b_1} \\
& \leq a_2 \cdot i^{b_2} \qquad \text{for suitable constants } a_2, b_2 \in \mathbb{R}_+ \\
& \stackrel{(2.8)}{\leq} a_2 \cdot (-A_2^{-1} \cdot \log(C_2^{-1} \cdot x_i))^{B_2^{-1} b_2} \\
& = (-a \cdot \log(c \cdot x_i))^b
\end{aligned}$$

for suitable constants  $a, b, c \in \mathbb{R}_+$  independent of  $i$ . This implies the claim as all other fibers of  $A$  are empty.  $\square$

We now deduce the desired more general version of uniform finiteness from Proposition 2.6.5. This even works for arbitrary d-minimal structures for which Proposition 2.6.5 holds. First, we prove a version for 0-dimensional families, then we generalize to arbitrary families of finite subsets of  $I$ .

**Proposition 2.6.6.** *Let  $A \subset I \times I$  be a definable family of finite subsets of  $I$ , s.t.  $\dim A = 0$ . Let  $L := \lim(\pi_1(A))$ . Then there exist constants  $a, b, c \in \mathbb{R}_+$  s.t. for all  $r > 0$  and  $x \in I \setminus \bigcup_{x' \in L} (x' - r, x' + r)$  we have:*

$$\#A_x \leq (-a \cdot \log(c \cdot r))^b$$

*Proof.* Consider  $A' := \{(x, y) \in I \times I \mid \exists x' \in I : (x', y) \in A \text{ and } x = d(x', L)\}$ . Then for given  $r \in I$ ,

$$A'_r = \bigcup_{x \in \pi_1(A), d(x, L) = r} A_x.$$

This is finite as the fibers of  $A$  are and there can be only finitely many  $x \in \pi_1(A)$  with a given distance to  $L$  (otherwise there would exist a limit point of  $\pi_1(A)$  outside  $\lim(\pi_1(A))$ ). Moreover,  $\pi_1(A')$  has no limit point other than 0, otherwise there were infinitely many elements of  $\pi_1(A)$  with distance  $\geq \varepsilon$  to  $\lim(\pi_1(A))$  for some  $\varepsilon \in \mathbb{R}_+$ , a contradiction. If 0 also wasn't a limit point of  $\pi_1(A')$ ,  $A$  would be finite, the claim then holds trivially. So we may assume that  $\pi_1(A')$  has unique limit point 0. We apply Proposition 2.6.5 to  $A'$  and obtain constants  $a, b, c \in \mathbb{R}_+$  s.t. for all  $x \in I$

$$\#A'_x \leq (-a \cdot \log(c \cdot x))^b.$$

Now let  $r \in \mathbb{R}_+$  be arbitrary,  $x \in I \setminus \bigcup_{x' \in L} (x' - r, x' + r)$ . Set  $r' := d(x, L)$ . Then  $r' \geq r$  and

$$\begin{aligned}
\#A_x & \leq \#A'_{r'} \\
& \leq (-a \cdot \log(c \cdot r'))^b \\
& \leq (-a \cdot \log(c \cdot r))^b.
\end{aligned}$$

This shows the claim.  $\square$

*Proof of Theorem 2.6.4.* Assume that we have constructed a definable, countable set  $Z' \subset [0, 1]$  s.t. the cardinality of the fibers of  $A$  is constant on any open interval disjoint from  $Z'$  (the construction is postponed). Then the same holds for  $\bar{Z} := \text{cl}(Z') \cup \{0, 1\}$ . Let  $Z$  be the union of  $\bar{Z}$  and all the midpoints between two neighbouring elements of  $\bar{Z}$  (this is definable).

Then  $Z \subset I$  has the following property:

**Claim.** For each  $x \in I \setminus Z$  there is a  $z \in Z$  s.t.  $\#A_x = \#A_z$  and  $d(z, \lim(Z)) \geq d(x, z)$ .

*Proof of Claim.* If  $x \notin Z$ , then  $x \notin \bar{Z}$  and, as  $\bar{Z}$  is closed and  $0, 1 \in \bar{Z}$ , there exist  $a, b \in \bar{Z}$  with  $x \in (a, b)$  and  $(a, b) \cap \bar{Z} = \emptyset$ . Then take  $z := \frac{a+b}{2} \in Z$ , as the cardinality of the fibers of  $A$  is constant on  $(a, b)$ , we have  $\#A_x = \#A_z$ . Moreover,  $z$  has distance at least  $\frac{b-a}{2}$  to all other elements of  $Z$ , implying  $d(z, \lim(Z)) \geq \frac{b-a}{2} \geq d(x, z)$ .  $\square$

Let  $A_Z$  be the subset of  $A$  with parameter coordinate in  $Z$ , i.e.  $A_Z := A \cap (Z \times I)$ . Then  $\pi_1(A_Z) \subset Z$  is countable, equivalently  $\dim A_Z = 0$ . Set  $L := \lim(\pi_1(A_Z)) \subset \lim(Z)$  (clearly 0-dimensional). By Proposition 2.6.6 there exist constants  $a, b, c \in \mathbb{R}_+$  s.t. for all  $r > 0$  and  $x \in I \setminus \bigcup_{x' \in L} (x' - \frac{r}{2}, x' + \frac{r}{2})$  we have:

$$\#(A_Z)_x \leq (-a \cdot \log(c \cdot r))^b. \quad (2.11)$$

Now let  $x \in I \setminus \bigcup_{x' \in L} (x' - r, x' + r)$ , we have to show  $\#A_x \leq (-a \cdot \log(c \cdot r))^b$ .

If  $x \in Z$ ,  $A_x = (A_Z)_x$  and we get the desired inequality directly by (2.11), so assume  $x \notin Z$ . By the claim there is  $z \in Z$  s.t.  $\#A_x = \#A_z$  and  $d(z, Z_{\lim}) \geq d(x, z)$ . As  $L \subset Z_{\lim}$ , we know  $d(z, L) \geq d(z, Z_{\lim})$ , hence  $d(z, L) \geq d(x, z)$ , implying

$$r \leq d(x, L) \leq d(x, z) + d(z, L) \leq 2 \cdot d(z, L),$$

using triangular inequality in the second inequality. So  $d(z, L) \geq \frac{r}{2}$  and, using (2.11):

$$\#A_x = \#A_z = \#(A_Z)_z \stackrel{(2.11)}{\leq} (-a \cdot \log(c \cdot r))^b,$$

proving the theorem.

Left to do is the construction of a definable, countable  $Z' \subset [0, 1]$  s.t. the cardinality of the corresponding fibers of  $A$  is constant on any open interval disjoint from  $Z'$ . Let

$$B := \{(x, y) \in \text{cl}(A) \mid \text{there is no } r \in \mathbb{R}_+ \text{ s.t. for every } x' \in (x - r, x + r) \text{ the set } \text{cl}(A) \cap (\{x'\} \times (y - r, y + r)) \text{ has exactly one element}\}.$$

$B$  can be thought of as the set of elements of  $\text{cl}(A)$  around which  $\text{cl}(A)$  locally does not look like the graph of a function. Set  $Z' := \pi_1(B)$ , this certainly is definable.

We first argue that  $Z'$  is countable: If  $Z'$  was uncountable, by countable cell decomposition applied to  $B$ , there must be a  $(1, 0)$ -cell contained in  $B$ , say  $\mathcal{C}$ . Now, by definition of  $B$ , for each point  $a$  in  $\mathcal{C}$  and any  $r \in \mathbb{R}_+$  there must be a point  $a' \in \text{cl}(A) \setminus \mathcal{C}$  which lies inside the box of radius  $r$  around  $a$ . This implies  $\mathcal{C} \subset \text{cl}(\text{cl}(A) \setminus \mathcal{C}) = \text{cl}(A \setminus \mathcal{C})$ . Now  $\bar{A} := A \setminus \mathcal{C}$  is a definable family of finite sets and [26, §7, Main Lemma] yields that  $\{x \in I \mid \text{cl}(\bar{A})_x \neq \bar{A}_x\}$  is countable. But by the above,  $\pi_1(\mathcal{C}) \subset \{x \in I \mid \text{cl}(\bar{A})_x \neq \bar{A}_x\}$ , a contradiction.

Finally we show that the cardinality of the corresponding fibers of  $A$  is constant on any open interval disjoint from  $Z'$ : For  $x \in I \setminus Z'$ , there is for each element  $y \in A_x$  an  $r \in \mathbb{R}_+$  s.t. for every  $x' \in (x - r, x + r)$  the set  $\text{cl}(A) \cap (\{x'\} \times (y - r, y + r))$  has exactly one element. Taking  $r$  small enough, this implies, that for every  $x' \in (x - r, x + r)$ ,  $\#A_{x'} = \#A_x$ . Hence the cardinality of the corresponding fibers of  $A$  is constant between two neighbouring elements of  $Z'$ .  $\square$

## Chapter 3

# A Countable Parametrization Theorem

The Pila-Wilkie theorem on counting rational points of definable sets is a major result connecting o-minimality with diophantine geometry. It has given a boost to the recognition of o-minimality since its publication in 2006 [33]. However, until now, it is not clear whether the theorem might be generalized to tame settings beyond o-minimality.

A main ingredient of the proof of the Pila-Wilkie theorem is the parametrization theorem. As a first step towards generalizing the Pila-Wilkie theorem to d-minimal structures, in this chapter we prove a d-minimal version of the parametrization theorem, a countable parametrization theorem.

In the first section we introduce the Pila-Wilkie theorem as well as the original parametrization theorem. Thereafter, we state the countable parametrization theorem for d-minimal structures which we prove in Sections 3.3 - 3.5. Finally, we comment on the necessities in order to deduce a d-minimal version of the Pila-Wilkie theorem.

### 3.1 The Pila-Wilkie theorem and the parametrization theorem

The Pila-Wilkie theorem (also called the counting theorem) has become one of the main examples of theorems in model theory that have deep consequences for other mathematical disciplines, in this case diophantine geometry. It deals with counting rational points in definable sets in o-minimal structures.

While algebraic (or semialgebraic) sets may contain arbitrarily many rational points, Pila and Wilkie showed in 2006 that the number of rational points in the 'transcendental' part of a definable set is bounded in a certain way [33]. We will make this precise below.

Pila and Wilkie's result was based on earlier work of Bombieri and Pila on counting integer points on graphs of transcendental real-analytic functions  $f : [0, 1] \rightarrow \mathbb{R}$  [4]. Trying to generalize the result there to higher dimensions, the transcendental part of compact subanalytic sets became the object of study [31]. Such sets are globally subanalytic, i.e. definable in the o-minimal structure  $\mathbb{R}_{\text{an}}$  (see Section 1.1). Finally, it turned out that o-minimal structures provide a natural setting for the desired kind of results, model theory entered the so-far diophantine arguments, and the Pila-Wilkie theorem in its current form

could be proven.

The Pila-Wilkie theorem has found elegant applications in arithmetic geometry, bringing to consciousness the topic of o-minimality to many number theorists. In 2008, Pila and Zannier gave a reproof of the Manin-Mumford conjecture using a completely new approach with the Pila-Wilkie theorem at the heart of the proof [34]. The method they used has become archetypical for later applications of the Pila-Wilkie theorem and is now known as the Pila-Zannier method. A major result was Pila's unconditional proof of a special version of the so-called André-Oort conjecture from 2011 (earning him the Clay Research Award) [32]. Recently, in 2021, Pila, Shankar and Tsimerman even succeeded in proving the André-Oort conjecture in full generality, essentially following the Pila-Zannier method [35].

In order to state the theorem we need to introduce some notation:

**Definition 3.1.1.** Denote by  $H : \mathbb{Q} \rightarrow \mathbb{N}_+$  the **height function** given by  $H(\frac{a}{b}) = \max\{|a|, |b|\}$  for coprime  $a, b \in \mathbb{Z}$ ,  $b \neq 0$ . For  $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$  set  $H(q) := \max\{H(q_1), \dots, H(q_n)\}$  and call  $H(q)$  the **height** of  $q$ .

For  $X \subset \mathbb{R}^n$  set  $X(\mathbb{Q}) := X \cap \mathbb{Q}^n$ , the rational points of  $X$ . For  $T \in \mathbb{R}$ ,  $T \geq 1$ , set  $X(\mathbb{Q}, T) := \{q \in X(\mathbb{Q}) \mid H(q) \leq T\}$  and  $N(X, T) := \#X(\mathbb{Q}, T)$ . Note that, in contrast to  $X(\mathbb{Q})$ ,  $X(\mathbb{Q}, T)$  is always finite as there are only finitely many rational numbers bounded by a given height.

The aim of the Pila-Wilkie theorem is to bound  $N(X, T)$  for suitable  $X$ . The underlying idea is that 'transcendental' sets should contain 'few' rational points.

**Definition 3.1.2.** Let  $X \subset \mathbb{R}^n$ . The **algebraic part** of  $X$ , denoted by  $X^{\text{alg}}$  is given by the union of all connected infinite semialgebraic subsets of  $X$ . The **transcendental part** of  $X$  is its complement  $X^{\text{trans}} := X \setminus X^{\text{alg}}$ .

The algebraic part of a set is often hard to identify and can become quite complex.

We can now state the Pila-Wilkie theorem:

**Theorem 3.1.3** (Pila-Wilkie Theorem, [33, Theorem 1.8]). *Let  $X \subset \mathbb{R}^n$  be definable in some o-minimal expansion of the real field. Then for all  $\varepsilon \in \mathbb{R}_+$  there is a constant  $C \in \mathbb{R}_+$  s.t. for all  $T \in \mathbb{R}$ ,  $T \geq 1$ :*

$$N(X^{\text{trans}}, T) \leq C \cdot T^\varepsilon$$

In fact, Pila and Wilkie prove the following stronger result:

**Theorem 3.1.4** (Pila and Wilkie, [33, Theorem 1.10]). *Fix an o-minimal expansion of the real field. Let  $X \subset \mathbb{R}^m \times \mathbb{R}^n$  be a definable family of subsets of  $\mathbb{R}^n$ . Let  $\varepsilon \in \mathbb{R}_+$ . Then there exists a definable family  $V \subset X$  and a constant  $C \in \mathbb{R}_+$  s.t. for all  $s \in \mathbb{R}^m$  we have  $V_s \subset X_s^{\text{alg}}$  and for all  $T \in \mathbb{R}$ ,  $T \geq 1$ :*

$$N(X_s \setminus V_s, T) \leq C \cdot T^\varepsilon.$$

The proof uses two main ingredients:

1. a result from Pila's prior work on counting integer points (Theorem 3.1.5)
2. the so-called parametrization theorem (Theorem 3.1.7)

From these two one easily deduces that the rational points in question lie on 'few' hypersurfaces of suitable degree. Then one establishes the desired bound proceeding by induction on the dimension of  $X$ .

A very detailed and comprehensive presentation of the complete proof of the Pila-Wilkie theorem, which also simplifies parts of it, was recently given by Bhardwaj and van den Dries [1].

We take a closer look at the two ingredients. The first can be easily derived from Pila's prior work [30, Proposition 4.2] and has nothing to do with o-minimality:

**Theorem 3.1.5** (Pila, [33, Proposition 6.1]). *Let  $n \in \mathbb{N}_+$ . Then, for each  $d \in \mathbb{N}_+$  there are  $k = k(n, d) \in \mathbb{N}$ ,  $\varepsilon = \varepsilon(n, d) \in \mathbb{R}_+$  and  $C = C(n, d) \in \mathbb{R}_+$  s.t. the following holds: Let  $m \in \mathbb{N}$ ,  $m < n$ , and  $\varphi : I^m \rightarrow \mathbb{R}^n$  a  $C^k$ -map with  $|\varphi^{(\mu)}(x)| \leq 1$  for all  $x \in I^m$ ,  $\mu \in \mathbb{N}^m$  with  $|\mu| \leq k$ . Let  $X = \text{im}(\varphi) \subset \mathbb{R}^n$ . Then for all  $T \in \mathbb{R}$  with  $T \geq 1$ ,  $X(\mathbb{Q}, T)$  is contained in the union of at most  $C \cdot T^\varepsilon$  hypersurfaces of degree  $\leq d$ . Moreover,  $\varepsilon(n, d) \rightarrow 0$  for  $d \rightarrow \infty$ .*

Roughly speaking, this says that one has good control about the number of hypersurfaces in which the rational points of the image of a sufficiently non-oscillatory map reside. Complementary, the parametrization theorem states that any bounded set definable in an o-minimal expansion of the real field is the finite union of images of such functions. Pila and Wilkie need this in a version for more general o-minimal structures, not only over the reals. We define:

**Definition 3.1.6.** Let  $\mathcal{R}$  be an o-minimal structure expanding an ordered field  $R$ . Let  $X \subset R^n$  be definable,  $k \in \mathbb{N}$ .

- $X$  is called **strongly bounded**, if there is a natural number  $N \in \mathbb{N}$  s.t. the absolute values of all coordinates of all elements of  $X$  are bounded by  $N$ . A definable function  $X \rightarrow R^m$  is **strongly bounded**, if its graph is so, equivalently if its domain and its image are strongly bounded.
- A **partial  $k$ -parametrization** of  $X$  is a definable  $C^k$ -map  $\varphi : I^{\dim X} \rightarrow X$  s.t.  $\varphi^{(\mu)}$  is strongly bounded for all  $\mu \in \mathbb{N}^{\dim X}$  with  $|\mu| \leq k$ .
- A  **$k$ -parametrization** of  $X$  is a finite set of partial  $k$ -parametrizations of  $X$  whose images cover  $X$ .
- A  **$k$ -reparametrization** of a definable map  $f : X \rightarrow R^m$  is a  $k$ -parametrization  $\Phi$  of  $X$  s.t. for every  $\varphi \in \Phi$ ,  $f \circ \varphi$  is of class  $C^k$  and  $(f \circ \varphi)^{(\mu)}$  is strongly bounded for all  $\mu \in \mathbb{N}^{\dim X}$  with  $|\mu| \leq k$ .

The parametrization theorem then takes the following form:

**Theorem 3.1.7** (Parametrization Theorem, Pila and Wilkie, [33, Theorems 2.3 and 2.5]). *Work in an o-minimal expansion of an ordered field. Let  $k \in \mathbb{N}$ . Then any strongly bounded, definable set  $X$  admits a  $k$ -parametrization. Any strongly bounded, definable function  $f$  admits a  $k$ -reparametrization.*

This theorem and its proof is based on work of Gromov who considered the semialgebraic case [22] (see [5] for a better presentation). Gromov himself refined a method of Yomdin [42], that is why the parametrization theorem is also known as the Yomdin-Gromov algebraic lemma.

The bound in the Pila-Wilkie theorem cannot be improved in general. In particular, it is optimal for the o-minimal structure  $\mathbb{R}_{\text{an}}$ . However, it was conjectured by Wilkie that for sets  $X$  definable in  $(\overline{\mathbb{R}}, \exp)$  we have

$$N(X^{\text{trans}}, T) \leq c_1 \cdot (\log T)^{c_2}$$

for suitable constants  $c_1, c_2 \in \mathbb{R}_+$  (depending on  $X$ ). This has recently been proven in 2022 by Binyamini, Novikov and Zack [3].

### 3.2 A countable parametrization theorem

As a first step towards generalizing the Pila-Wilkie theorem to the d-minimal setting, the goal of this chapter is to prove a d-minimal version of the parametrization theorem.

Certainly, the original theorem doesn't hold in the d-minimal case: Subsets of  $\mathbb{R}^n$  with infinitely many connected components, such as  $2^{\mathbb{Z}}$ , cannot be parametrized by *finitely* many partial parametrizations (as the image of a partial parametrization is necessarily connected). However, the situation changes if we allow for *countable* parametrizations.

This resembles the different versions of cell decomposition discussed above: In the o-minimal world, any definable set is the union of *finitely* many cells while in the d-minimal setting the same holds with *countably* many cells. This is not by coincidence: In fact, the proof of the countable parametrization theorem given below heavily relies on countable cell decomposition.

In this section we give the necessary definitions and state the countable parametrization theorem. As countable cell decomposition as well as several other theorems in d-minimality are proven only for structures over  $\mathbb{R}$ , we restrict ourselves to working in expansions of the real field.

For the rest of this chapter, fix a d-minimal expansion  $\mathcal{R}$  of  $\overline{\mathbb{R}}$ .

Most parts of our proof consist of adjusting Binyamini and Novikov's proof of the parametrization theorem [2]. They simplified Pila and Wilkie's proof and thereby made it more suitable for an adaptation to the d-minimal setting. A major difference to the original proof is the use of so-called *cellular* parametrizations:

**Definition 3.2.1.** A **basic cell**  $\mathcal{C} \subset \mathbb{R}^m$  of **length**  $m$  is a product  $\prod_{i=1}^m \mathcal{C}_i$  where each  $\mathcal{C}_i$  is either  $I$  or  $\{0\}$ . A continuous map  $f = (f_1, \dots, f_m) : \mathcal{C} \rightarrow \mathbb{R}^m$  is called **cellular** if for every  $i = 1, \dots, m$

- (i)  $f_i$  depends only on the first  $i$  coordinates, and
- (ii)  $f_i$  is strictly increasing in the  $i$ -th variable.

Note in particular that cellular maps preserve dimension and the composition of cellular maps is cellular.

A map  $f : \mathcal{C} \rightarrow \mathbb{R}^m$  is **of class**  $C^k$ , if all the partial derivatives in the directions  $x_i$  with  $\mathcal{C}_i = I$  up to order  $k$  exist and are continuous. In this case, for  $\mu \in \mathbb{N}^m$ ,  $|\mu| \leq k$ , we denote by  $f^{(\mu)}$  the corresponding partial derivative with  $f^{(\mu)} = 0$  if  $\mu_j > 0$  for some  $j$  with  $\mathcal{C}_j = \{0\}$ .

**Definition 3.2.2.** Let  $X \subset \mathbb{R}^m$  be definable.

- $X$  is called **bounded** if  $X \subset [-N, N]^m$  for some  $N \in \mathbb{N}$ .
- A definable map  $f : X \rightarrow \mathbb{R}^n$  is called **bounded** if its graph  $\Gamma(f) \subset \mathbb{R}^{m+n}$  is so; equivalently, its domain and its image are bounded.
- A **partial cellular  $k$ -parametrization** of  $X$  is a definable cellular  $C^k$ -map  $\varphi : \mathcal{C} \rightarrow X$  for some basic cell  $\mathcal{C}$  of length  $m$  such that  $\varphi^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^m$  with  $|\mu| \leq k$ .
- A **countable cellular  $k$ -parametrization** of  $X$  is a countable set of partial cellular  $k$ -parametrizations of  $X$  whose images cover  $X$ .
- A **countable cellular  $k$ -reparametrization** of a definable map  $f : X \rightarrow \mathbb{R}^n$  is a countable cellular  $k$ -parametrization  $\Phi$  of its domain  $X$  such that for every  $\varphi \in \Phi$ ,  $f \circ \varphi$  is of class  $C^k$  and  $(f \circ \varphi)^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^m$  with  $|\mu| \leq k$ .

**Remark 3.2.3.** In the following, we sometimes omit 'countable cellular' and just say '(partial)  $k$ -parametrization' or ' $k$ -reparametrization'. We will not refer to the definition of parametrization from the previous section again.

We will often write a  $k$ -parametrization of a set  $X$  as  $\Phi = \{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow X\}$ , in these cases  $\alpha$  is implicitly assumed to range over a countable index set.

We can now state the countable parametrization theorem.

**Theorem 3.2.4** (Countable Parametrization Theorem). *Let  $k \in \mathbb{N}$ . Any bounded definable set  $X$  admits a countable cellular  $k$ -parametrization.*

*Any bounded definable map  $f : X \rightarrow \mathbb{R}^n$  admits a countable cellular  $k$ -reparametrization.*

We finish this section by proving that for reparametrization we only need to consider the case  $n = 1$ :

**Lemma 3.2.5.** *Let  $k, m \in \mathbb{N}$  and suppose that every bounded definable function  $X \rightarrow \mathbb{R}$  with  $X \subset \mathbb{R}^m$  has a countable cellular  $k$ -reparametrization. Then every bounded definable map  $X \rightarrow \mathbb{R}^n$  with  $X \subset \mathbb{R}^m$ ,  $n \in \mathbb{N}$  has a countable cellular  $k$ -reparametrization.*

*Proof.* Let  $n \in \mathbb{N}$  and suppose  $F : X \rightarrow \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}$  with  $X \subset \mathbb{R}^m$  are definable and bounded and  $F$  has a  $k$ -reparametrization. By induction, it is enough to show that then the definable bounded map  $(F, f) : X \rightarrow \mathbb{R}^{n+1}$  has a  $k$ -reparametrization. Let  $\Phi = \{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow X\}$  be a  $k$ -reparametrization of  $F$ . For each  $\alpha$  we can apply the hypothesis of the lemma to obtain a  $k$ -reparametrization  $\Psi_\alpha$  of the function  $f \circ \varphi_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R}$ . Then  $\{\varphi_\alpha \circ \psi : \varphi_\alpha \in \Phi, \psi \in \Psi_\alpha\}$  is a countable cellular  $k$ -reparametrization of  $(F, f)$  by the chain rule.  $\square$

### 3.3 Proof of the countable parametrization theorem

We prove parametrization and reparametrization simultaneously by induction. We consider the following statements depending on  $m \in \mathbb{N}$ :

- (I) <sub>$m$</sub>  For all  $k \in \mathbb{N}$ , every bounded definable set  $X \subset \mathbb{R}^m$  has a countable cellular  $k$ -parametrization.
- (II) <sub>$m$</sub>  For all  $k, n \in \mathbb{N}$ , every bounded definable map  $f : X \rightarrow \mathbb{R}^n$  with  $X \subset \mathbb{R}^m$  has a countable cellular  $k$ -reparametrization.

It is clear that (I)<sub>0</sub> and (II)<sub>0</sub> hold. In the following section we show the base case (II)<sub>1</sub>. Thereafter we continue proving the first induction step (I)<sub>m</sub>+(II)<sub>m</sub> ⇒ (I)<sub>m+1</sub> (for all  $m \in \mathbb{N}$ ). Finally, we give the proof of the second induction step (I)<sub>m+1</sub>+(II)<sub>m</sub> ⇒ (II)<sub>m+1</sub> (for all  $m \in \mathbb{N}_+$ ).

All results and proofs in this section are based on Binyamini and Novikov's presentation for the o-minimal case [2].

We make abundant use of the following observation about composing parametrizations:

**Remark 3.3.1.** Assume that  $\Phi = \{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow X\}$  is a  $k$ -parametrization of some definable set  $X$  and, for each  $\alpha$ ,  $\Phi_\alpha = \{\varphi_{\alpha,\beta} : \mathcal{C}_{\alpha,\beta} \rightarrow \mathcal{C}_\alpha\}$  is a  $k$ -parametrization of  $\mathcal{C}_\alpha$ . Then the collection  $\{\varphi_\alpha \circ \varphi_{\alpha,\beta}\}$  is a  $k$ -parametrization of  $X$  by the chain rule.

Moreover, if  $f : X \rightarrow \mathbb{R}^n$  is a definable map,  $\Phi = \{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow X\}$  is a  $k$ -parametrization of  $X$  and, for each  $\alpha$ ,  $\Phi_\alpha = \{\varphi_{\alpha,\beta} : \mathcal{C}_{\alpha,\beta} \rightarrow \mathcal{C}_\alpha\}$  is a  $k$ -reparametrization of  $f \circ \varphi_\alpha$ , then  $\{\varphi_\alpha \circ \varphi_{\alpha,\beta}\}$  is a  $k$ -reparametrization of  $f : X \rightarrow \mathbb{R}^n$ .

### 3.3.1 Induction basis - reparametrizing unary functions

We have to prove that any bounded definable map  $f : X \rightarrow \mathbb{R}^n$  with  $X \subset \mathbb{R}$  has for any  $k \in \mathbb{N}$  a countable cellular  $k$ -reparametrization (we may already assume  $n = 1$  by Lemma 3.2.5).

We start with a special case whose proof already contains the crucial step of the argument: A simple analytic observation about composing maps with a quadratic function shows how to get the boundedness of the next derivative if one already has boundedness of the derivatives up to a given order.

**Lemma 3.3.2.** *Let  $f : I \rightarrow \mathbb{R}$  be a definable  $C^k$ -function,  $k \geq 2$ , with bounded  $f^{(j)}$  for  $0 \leq j \leq k-1$ . Assume moreover that  $|f^{(k)}|$  is decreasing. Then  $f$  has a cellular countable  $k$ -reparametrization (which moreover consists of only one partial parametrization).*

*Proof.* Let  $\varphi : I \rightarrow I$  be given by  $\varphi(x) = x^2$ . Certainly,  $\varphi$  forms a cellular  $k$ -parametrization of  $I$ . It is left to show that  $(f \circ \varphi)^{(l)}$  is bounded for all  $l = 0, \dots, k$ . For  $l < k$  this holds by the assumptions on  $f$  and the chain rule. Computing  $(f \circ \varphi)^{(k)}$  with the chain rule, the only summand that is not immediately bounded by the assumptions is  $2^k x^k f^{(k)}(x^2)$ .

Let  $f^{(k-1)}$  be bounded by  $C \in \mathbb{R}$ . We claim that for all  $x \in I$  we have  $|f^{(k)}(x)| \leq 4C/x$ . This finishes the proof as then

$$|2^k x^k f^{(k)}(x^2)| \leq x^k \cdot 4C/x^2 \leq 4C,$$

using  $k \geq 2$ .

Assume for a contradiction that there was  $x_0 \in I$  such that  $|f^{(k)}(x_0)| > 4C/x_0$ . Then the Mean Value Theorem provides a  $\xi \in [x_0/2, x_0]$  such that

$$f^{(k-1)}(x_0) - f^{(k-1)}(x_0/2) = f^{(k)}(\xi) \cdot (x_0 - x_0/2) = f^{(k)}(\xi) \cdot x_0/2.$$

Since  $|f^{(k)}|$  is decreasing by assumption,  $|f^{(k)}(\xi)| \geq |f^{(k)}(x_0)| > 4C/x_0$ . Hence

$$2C \geq |f^{(k-1)}(x_0) - f^{(k-1)}(x_0/2)| > (4C/x_0) \cdot (x_0/2) = 2C,$$

a contradiction. □

The rest of the proof mainly is a careful application of the d-minimal structure theorems allowing to reduce to the proven special case, while ensuring that all parametrizing functions remain cellular. First, we remove the assumption that  $|f^{(k)}|$  is decreasing.



**Lemma 3.3.3.** *Let  $f : I \rightarrow \mathbb{R}$  be a definable  $C^k$ -function,  $k \geq 2$ , with bounded  $f^{(j)}$  for  $0 \leq j \leq k-1$ . Then  $f$  has a cellular countable  $k$ -reparametrization.*

*Proof.* By the d-minimal version of the monotonicity theorem we may decompose  $I$  into countably many points and countably many subintervals on which  $|f^{(k)}|$  is monotone. We can reparametrize on each of these subsets individually.

For the points in this decomposition, say  $p \in I$ , choose constant partial  $k$ -reparametrizations  $\{0\} \rightarrow \{p\}$ . For the subintervals, say  $(a, b)$ , we first take linear strictly increasing bijections  $\delta : I \rightarrow (a, b)$ , then by Remark 3.3.1 it suffices to find a  $k$ -reparametrization of  $f \circ \delta$ . Hence we can wlog assume that  $|f^{(k)}|$  is either decreasing or increasing.

Lemma 3.3.2 takes care of the first case. If  $|f^{(k)}|$  is increasing, we can apply Lemma 3.3.2 to  $f \circ \gamma$  where  $\gamma : I \rightarrow I$  sends  $x$  to  $1-x$ . This gives that  $\varphi : I \rightarrow I$ ,  $\varphi(x) = x^2$  is a  $k$ -reparametrization of  $f \circ \gamma$ , implying that  $\gamma \circ \varphi \circ \gamma$  is a cellular  $k$ -reparametrization of  $f$ .  $\square$

The next lemma is concerned with parametrizations of one-dimensional definable subsets of  $I^2$  and we will afterwards apply it to the graph of a definable bounded function. The reason for this change of perspective is that in this setting we may easier reduce to the case where the first derivative is bounded (by considering the inverse function, see Step 2 in the proof). This is necessary as Lemma 3.3.3 only applies if the first derivative is already bounded.

**Lemma 3.3.4.** *Let  $X \subset I^2$  be definable of dimension 1. For every  $k \in \mathbb{N}$  there exists a countable collection of definable  $C^k$ -maps  $\{\varphi_\alpha : I \rightarrow X\}$  such that:*

- (i)  $\bigcup_\alpha \varphi_\alpha(I) = X \setminus \Sigma$  for some countable set  $\Sigma$ .
- (ii)  $\varphi_\alpha^{(l)}$  is bounded for all  $l = 0, \dots, k$ .
- (iii) Every coordinate function of every  $\varphi_\alpha$  is either strictly monotone or constant.

*Proof.*

**Step 1.** *Assume  $X = \Gamma(f)$  for a definable  $C^k$ -function  $f : I \rightarrow I$  with strictly positive and bounded  $f'$ .*

We apply Lemma 3.3.3 repeatedly  $k-1$  times and get a countable cellular  $k$ -reparametrization  $\{\tilde{\varphi}_\alpha : \mathcal{C}_\alpha \rightarrow I\}$  of  $f$ . For each  $\alpha$  with  $\mathcal{C}_\alpha = \{0\}$  we add  $(\tilde{\varphi}_\alpha(0), f(\tilde{\varphi}_\alpha(0)))$  to  $\Sigma$  and for each  $\alpha$  with  $\mathcal{C}_\alpha = I$  we set  $\varphi_\alpha = (\tilde{\varphi}_\alpha, f \circ \tilde{\varphi}_\alpha) : I \rightarrow X$ . This gives a countable collection of definable  $C^k$ -maps which satisfies the first two conditions in the claim as the  $\tilde{\varphi}_\alpha$  form a  $k$ -reparametrization of  $f$ . The third condition is satisfied as  $f$  is strictly increasing by assumption and  $\tilde{\varphi}_\alpha$  is strictly increasing as it is cellular for each  $\alpha$ .

**Step 2.** *General case*

By countable cell decomposition, we can decompose  $X$  into countably many points, vertical intervals of the form  $\{x_0\} \times (a, b)$  ( $x_0, a, b \in I, a < b$ ) and graphs of definable functions  $f : (a, b) \rightarrow I$  ( $a, b \in I, a < b$ ). We denote by  $\Sigma$  the set of points. We need to find countably many functions  $\varphi_\alpha$  as in the claim whose images cover  $X \setminus \Sigma$ . We treat each of the 1-dimensional subsets of the above decomposition separately.

For a vertical interval of the form  $\{x_0\} \times (a, b)$  ( $x_0, a, b \in I, a < b$ ), we just take  $\varphi : I \rightarrow X$  given by  $\varphi(x) = (x_0, a + x \cdot (b - a))$ . This certainly satisfies the requirements.

Now let  $f : (a, b) \rightarrow I$  be a definable function ( $a, b \in I, a < b$ ). By d-minimal generic

smoothness and the d-minimal monotonicity theorem together with countable cell decomposition, we can partition  $(a, b)$  into countably many points and subintervals on which  $f$  is either constant or monotone,  $C^k$ -smooth (or at least  $C^1$  if  $k = 0$ ), and one of

$$f' \leq -1 \quad -1 \leq f' < 0 \quad 0 < f' \leq 1 \quad 1 \leq f'$$

holds. We partition the graph of  $f$  accordingly and treat each of the subsets individually. The points we add to  $\Sigma$ . For the graph of a constant function  $f : (c, d) \rightarrow I$  with value  $y_0$  we can simply take  $\varphi : I \rightarrow X$  given by  $\varphi(x) = (c + x \cdot (d - c), y_0)$ . This certainly satisfies the requirements.

Finally, let  $X$  be the graph of a  $C^k$ -function  $f : (c, d) \rightarrow I$  for which one of the above relations holds. If  $|f'| \geq 1$ , we may replace  $X$  by  $\tau(X)$  with  $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  swapping coordinates. Certainly, a solution for  $\tau(X)$  will immediately give a solution for  $X$ .  $\tau(X)$  is the graph of the definable  $C^k$ -function  $f^{-1} : f((c, d)) \rightarrow I$  (note that  $f((c, d))$  must be a subinterval of  $I$ ) for which we have  $|(f^{-1})'| \leq 1$  as  $|f'| \geq 1$ . Replacing  $f$  by  $f^{-1}$  we have reduced to the case  $|f'| \leq 1$ .

Assuming  $-1 \leq f' < 0$ , we may replace  $X$  by  $\rho(X)$  where  $\rho : I^2 \rightarrow I^2$  is given by  $\rho(x, y) = (1 - x, y)$ . Certainly, a solution for  $\rho(X)$  will immediately give a solution for  $X$ .  $\rho(X)$  is the graph of the definable  $C^k$ -function  $f \circ \gamma : (1 - b, 1 - a) \rightarrow I$  where  $\gamma : I \rightarrow I$  sends  $x$  to  $1 - x$ . Now, as  $-1 \leq f' < 0$ , we have  $0 < (f \circ \gamma)' \leq 1$ . Replacing  $f$  by  $f \circ \gamma$  we have reduced to the case  $0 < f' \leq 1$ .

Rescaling to  $I$  as in the proof of Lemma 3.3.3 reduces to the case considered in Step 1.  $\square$

Finally we get:

**Lemma 3.3.5.** *Let  $f : X \rightarrow \mathbb{R}$  be a bounded definable function,  $X \subset \mathbb{R}$ ,  $k \in \mathbb{N}$ . Then  $f$  has a countable cellular  $k$ -parametrization.*

*Proof.* After rescaling, we may assume that the graph of  $f$  is contained in  $I^2$ . Apply Lemma 3.3.4 to the graph of  $f$  and let  $\{\varphi_\alpha = (\varphi_\alpha^x, \varphi_\alpha^y) : I \rightarrow \Gamma(f)\}$  denote the resulting countable collection,  $\Sigma = \Gamma(f) \setminus \bigcup_\alpha \varphi_\alpha(I)$  countable by the first property in Lemma 3.3.4. Now for each  $\alpha$  let

$$\tilde{\varphi}_\alpha := \begin{cases} \varphi_\alpha^x : I \rightarrow X & \text{if } \varphi_\alpha^x \text{ is strictly increasing} \\ \varphi_\alpha^x \circ \gamma : I \rightarrow X & \text{if } \varphi_\alpha^x \text{ is strictly decreasing,} \\ c : \{0\} \rightarrow X & \text{if } \varphi_\alpha^x \text{ is constantly } c \end{cases}$$

where  $\gamma : I \rightarrow I$  again denotes the function sending  $x$  to  $1 - x$ . This is well-defined by the third property in Lemma 3.3.4 and ensures that all the  $\tilde{\varphi}_\alpha$  are cellular maps. Now consider the countable collection

$$\Phi = \{\tilde{\varphi}_\alpha^x\} \cup \{x_0 : \{0\} \rightarrow X \mid (x_0, y_0) \in \Sigma\}$$

Then  $\Phi$  forms a countable cellular  $k$ -reparametrization of  $f$ :

Each map in  $\Phi$  is definable, cellular and  $C^k$  and the images cover  $X$  by construction. Moreover,  $\varphi^{(l)}$  is bounded for any  $\varphi \in \Phi$  and  $l = 0, \dots, k$  by the second property of Lemma 3.3.4. The same holds for  $(f \circ \varphi)^{(l)}$  using  $f \circ \varphi_\alpha^x = \varphi_\alpha^y$  and again the second property of Lemma 3.3.4.  $\square$

By Lemma 3.2.5 we have proven the unary case of the countable reparametrization theorem.

### 3.3.2 First induction step

Recall the statements we have to prove:

- (I)<sub>m</sub> For all  $k \in \mathbb{N}$ , every bounded definable set  $X \subset \mathbb{R}^m$  has a countable cellular  $k$ -parametrization.
- (II)<sub>m</sub> For all  $k, n \in \mathbb{N}$ , every bounded definable map  $f : X \rightarrow \mathbb{R}^n$  with  $X \subset \mathbb{R}^m$  has a countable cellular  $k$ -reparametrization.

In this subsection we prove  $(I)_m + (II)_m \Rightarrow (I)_{m+1}$  for all  $m \in \mathbb{N}$ . The proof crucially uses countable cell decomposition and reveals why we consider reparametrization of functions and not only parametrization of sets.

*Proof of  $(I)_m + (II)_m \Rightarrow (I)_{m+1}$ .* Let  $m \in \mathbb{N}$ . Assume  $(I)_m$  and  $(II)_m$ . Let  $k \in \mathbb{N}$  and  $X \subset \mathbb{R}^{m+1}$  be bounded definable. We have to show that  $X$  admits a countable cellular  $k$ -parametrization. By countable cell decomposition, we can decompose  $X$  into countably many cells and treat each separately, hence we can wlog assume that  $X$  is a cell.

**Case 1.**  $X = (f, g)_Y$  where  $Y$  is a bounded cell in  $\mathbb{R}^m$  and  $f, g : Y \rightarrow \mathbb{R}$  are bounded continuous definable functions with  $f(y) < g(y)$  for all  $y \in Y$ . Consider the function  $(f, g) : Y \rightarrow \mathbb{R}^2$ . By  $(II)_m$ , it has a countable cellular  $k$ -reparametrization, say  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow Y\}$ . For each  $\alpha$  let  $\psi_\alpha : \mathcal{C}_\alpha \times I \rightarrow X$  be given by

$$\psi_\alpha(s, t) = (\varphi_\alpha(s), t \cdot (g \circ \varphi_\alpha)(s) + (1 - t) \cdot (f \circ \varphi_\alpha)(s)).$$

Then  $\{\psi_\alpha : \mathcal{C}_\alpha \times I \rightarrow X\}$  forms a countable cellular  $k$ -parametrization of  $X$ .

**Case 2.**  $X = \Gamma(f)$  where  $f : Y \rightarrow \mathbb{R}$  a bounded continuous definable function on a cell  $Y \subset \mathbb{R}^m$ . By  $(II)_m$ ,  $f$  has a countable cellular  $k$ -reparametrization, say  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow Y\}$ . For each  $\alpha$  let  $\psi_\alpha : \mathcal{C}_\alpha \times \{0\} \rightarrow X$  be given by

$$\psi_\alpha(s, 0) = (\varphi_\alpha(s), (f \circ \varphi_\alpha)(s)).$$

Then  $\{\psi_\alpha : \mathcal{C}_\alpha \times \{0\} \rightarrow X\}$  forms a countable cellular  $k$ -parametrization of  $X$ .  $\square$

### 3.3.3 Second induction step

Finally, assume  $(I)_{m+1}$  and  $(II)_m$  for some  $m \in \mathbb{N}_+$ . The goal is to prove  $(II)_{m+1}$ .

Let  $k, n \in \mathbb{N}$ ,  $f : X \rightarrow \mathbb{R}^n$  a bounded definable map with  $X \subset \mathbb{R}^{m+1}$ . We have to show that  $f$  admits a countable cellular  $k$ -reparametrization. We start by several reduction steps, exploiting the induction hypotheses.

**Reduction Step 1.** *Reduction to  $f : I^{m+1} \rightarrow \mathbb{R}$ .*

*Proof.* By Lemma 3.2.5 we can wlog assume  $n = 1$ . Moreover, by  $(I)_{m+1}$ , we know that there is a countable cellular  $k$ -parametrization of  $X$ . Using Remark 3.3.1 this allows for reducing to the case that  $X$  is a basic cell of length  $m + 1$ . If  $X$  has any  $\{0\}$ -coordinates, we can use  $(II)_m$  to get the desired. Hence we can wlog assume that  $X = I^{m+1}$ .  $\square$

In order to simplify notation for the next reduction step, we introduce the following definition.

**Definition 3.3.6.** Let  $f : X \rightarrow \mathbb{R}$  a map with  $X \subset \mathbb{R}^{m+1}$ . Then we say  $f$  is **of class  $C^k$  in the last  $m$  variables** if for any fixed  $x_1 \in \pi_1(X)$  (with  $\pi_1 : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$  the projection on the first coordinate),  $X_{x_1} = \{(x_2, \dots, x_{m+1}) \mid (x_1, x_2, \dots, x_{m+1}) \in X\} \subset \mathbb{R}^m$  is a basic cell of length  $m$  and  $f(x_1, -) : X_{x_1} \rightarrow \mathbb{R}$  is of class  $C^k$ .

Then, for  $\mu \in \mathbb{N}^{m+1}$  with  $|\mu| \leq k$  and  $\mu_1 = 0$ , we denote by  $f^{(\mu)} : X \rightarrow \mathbb{R}$  the function sending  $(x_1, \dots, x_{m+1})$  to  $f(x_1, -)^{(\mu_2, \dots, \mu_{m+1})}(x_2, \dots, x_{m+1})$ . As usual,  $\frac{\partial f}{\partial x_i}$  stands for  $f^{(0, \dots, 0, 1, 0, \dots, 0)}$  with the 1 at the  $i$ -th position for  $i = 2, \dots, m+1$ .

Call  $f$   **$k$ -bounded in the last  $m$  variables** if  $f$  is of class  $C^k$  in the last  $m$  variables and  $f^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^{m+1}$  with  $|\mu| \leq k$  and  $\mu_1 = 0$ .

Now we come to see the whole point of using *cellular* parametrizations like Binyamini and Novikov instead of the less demanding notion of parametrization from the original paper of Pila and Wilkie: *Cellular* parametrizations guarantee that the property of being  $k$ -bounded in the last  $m$  variables is preserved under precomposition with a parametrization. This is the only point where we will use that a map is cellular.

**Lemma 3.3.7.** *Let  $f : X \rightarrow \mathbb{R}$  be a map with  $X \subset \mathbb{R}^{m+1}$ ,  $\varphi : \mathcal{C} \rightarrow X$  a partial cellular  $k$ -parametrization. Assume  $f$  is  $k$ -bounded in the last  $m$  variables. Then also  $f \circ \varphi$  is  $k$ -bounded in the last  $m$  variables.*

*Proof.* Denote by  $\varphi_1, \dots, \varphi_{m+1} : \mathcal{C} \rightarrow \mathbb{R}$  the coordinate functions of  $\varphi$ . As  $\varphi$  is cellular,  $\varphi_1$  does only depend on the first coordinate, hence we can regard it as a function  $\varphi_1 : \pi_1(\mathcal{C}) \rightarrow \mathbb{R}$  (with  $\pi_1$  the projection of the first coordinate). Then for fixed  $x_1 \in \pi_1(\mathcal{C})$ ,  $\mathcal{C}_{x_1}$  is a basic cell of length  $m$  and we have

$$(f \circ \varphi)(x_1, -) = f(\varphi_1(x_1), -) \circ (\varphi_2, \dots, \varphi_{m+1}) : \mathcal{C}_{x_1} \rightarrow \mathbb{R}. \quad (3.1)$$

Now  $(\varphi_2, \dots, \varphi_{m+1})$  is of class  $C^k$  as  $\varphi$  is a partial parametrization and  $f(\varphi_1(x_1), -)$  is of class  $C^k$  by assumption on  $f$ , hence  $f \circ \varphi$  is of class  $C^k$  in the last  $m$  variables.

Left to show is that  $(f \circ \varphi)^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^{m+1}$  with  $|\mu| \leq k$  and  $\mu_1 = 0$ . Exemplarily, we show that  $\frac{\partial(f \circ \varphi)}{\partial x_2}$  is bounded. By the chain rule, using (3.1) and unwinding the involved definitions, we have

$$\frac{\partial(f \circ \varphi)}{\partial x_2} = \sum_{i=2}^{m+1} \left( \frac{\partial f}{\partial x_i} \circ \varphi \right) \cdot \frac{\partial \varphi_i}{\partial x_2}$$

Now  $\frac{\partial f}{\partial x_i}$  is bounded by assumption on  $f$  and  $\frac{\partial \varphi_i}{\partial x_2}$  is bounded as  $\varphi$  is a partial parametrization. Thus,  $\frac{\partial(f \circ \varphi)}{\partial x_2}$  is bounded and similarly this holds for all derivatives up to order  $k$  that do not include the  $x_1$ -direction.  $\square$

We are ready to continue the main proof with the next reduction step which crucially uses (a family version of) the induction hypothesis  $(\text{II})_m$ .

**Reduction Step 2.** *Reduction to  $f : I^{m+1} \rightarrow \mathbb{R}$   $k$ -bounded in the last  $m$  variables*

*Proof.* Assume that we have shown  $(\text{II})_{m+1}$  for functions of this kind. The goal is to prove that an arbitrary bounded definable map  $f : I^{m+1} \rightarrow \mathbb{R}$  has a countable cellular  $k$ -reparametrization. Note that the assumption readily implies that functions  $f : X \rightarrow \mathbb{R}$  with arbitrary domain  $X \subset I^{m+1}$  that are  $k$ -bounded in the last  $m$  variables admit a countable cellular  $k$ -reparametrization (by precomposing with a parametrization of  $X$  as in the first reduction step, using that by Lemma 3.3.7 the property of being  $k$ -bounded in the last  $m$  variables is preserved).

Consider the following family version of  $(\text{II})_m$ :

**Claim.** Let  $l \in \mathbb{N}$ ,  $\{f_s : X_s \rightarrow \mathbb{R}^n\}_{s \in I^l}$  a definable family of uniformly bounded functions with  $X_s \subset \mathbb{R}^m$ . Then for all  $k \in \mathbb{N}$ , there exist countably many definable families of functions  $\{\varphi_{\alpha,s} : \mathcal{C}_{\alpha,s} \rightarrow X_s\}_{s \in I^l}$  and basic cells  $\mathcal{C}_\alpha$  s.t.

- (i) For all  $\alpha$  and  $s \in I^l$ , either  $\mathcal{C}_{\alpha,s} = \mathcal{C}_\alpha$  or  $\mathcal{C}_{\alpha,s} = \emptyset$ .
- (ii) For fixed  $s \in I^l$ ,  $\{\varphi_{\alpha,s} : \mathcal{C}_{\alpha,s} \rightarrow X_s\}$  forms a  $k$ -reparametrization of  $f_s$  (ignoring the possibly empty functions).
- (iii) For fixed  $\alpha$ , the derivatives of  $\varphi_{\alpha,s}$  and  $f_s \circ \varphi_{\alpha,s}$  up to order  $k$  are even bounded uniformly in  $s$ .

This can be obtained with essentially the same proof as the absolute version by adding parameters to all of the statements in the current section. Thus, by conceding that we are actually proving the family version and just omit confusing notation, we may use the claim freely. More explanation on how the proof of the family version differs from the absolute one is given in Section 3.5.

We apply the family version of (II) $_m$  to our function  $f : I^{m+1} \rightarrow \mathbb{R}$ , thinking  $x_1$  as a parameter. This gives a countable collection of definable families  $\{\varphi_{\alpha,x_1} : \mathcal{C}_{\alpha,x_1} \rightarrow I^m\}_{x_1 \in I}$  s.t. for every  $x_1 \in I$   $\{\varphi_{\alpha,x_1} : \mathcal{C}_{\alpha,x_1} \rightarrow I^m\}$  is a countable cellular  $k$ -reparametrization of  $f(x_1, -) : I^m \rightarrow \mathbb{R}$ . Moreover, all the derivatives that need to be bounded for this, are even uniformly bounded (independent of  $x_1$ ).

For fixed  $\alpha$  define

- $I_\alpha := \{(x_1, y) \mid x_1 \in I, y \in \mathcal{C}_{\alpha,x_1}\} \subset I^{m+1}$ .
- $\varphi_\alpha : I_\alpha \rightarrow I^{m+1}$  given by  $\varphi_\alpha(x_1, y) = (x_1, \varphi_{\alpha,x_1}(y))$
- $f_\alpha := f \circ \varphi_\alpha : I_\alpha \rightarrow \mathbb{R}$ .

As the  $\varphi_{\alpha,x_1}$  are of class  $C^k$  and the derivatives of  $\varphi_{\alpha,x_1}$  up to order  $k$  are uniformly bounded, we know that the  $\varphi_\alpha$  are  $k$ -bounded in the last  $m$  variables. By our assumption the claim holds for these kind of functions, so we get for each  $\alpha$  a countable cellular  $k$ -reparametrization  $\{\psi_{\alpha,\beta} : \mathcal{C}_{\alpha,\beta} \rightarrow I_\alpha\}$  of  $\varphi_\alpha$ .

Moreover, for  $x_1 \in I$ , the derivatives of  $f(x_1, -) \circ \varphi_{\alpha,x_1}$  up to order  $k$  are bounded uniformly in  $x_1$  by the choice of the  $\varphi_{\alpha,x_1}$ . This implies that  $f_\alpha$  is  $k$ -bounded in the last  $m$  variables for any  $\alpha$ . The same necessarily holds for any  $f_\alpha \circ \psi_{\alpha,\beta}$  by Lemma 3.3.7.

Again by our assumption, we get a cellular countable reparametrization of  $f_\alpha \circ \psi_{\alpha,\beta}$ , say  $\{\xi_{\alpha,\beta,\gamma} : \mathcal{C}_{\alpha,\beta,\gamma} \rightarrow \mathcal{C}_{\alpha,\beta}\}$ . Essentially by Remark 3.3.1, the  $\varphi_\alpha \circ \psi_{\alpha,\beta} \circ \xi_{\alpha,\beta,\gamma} : \mathcal{C}_{\alpha,\beta,\gamma} \rightarrow I^{m+1}$  form a countable cellular reparametrization of  $f$ .

$$\begin{array}{ccccccc} \mathcal{C}_{\alpha,\beta,\gamma} & \xrightarrow{\xi_{\alpha,\beta,\gamma}} & \mathcal{C}_{\alpha,\beta} & \xrightarrow{\psi_{\alpha,\beta}} & I_\alpha & \xrightarrow{\varphi_\alpha} & I^{m+1} & \xrightarrow{f} & \mathbb{R} \\ & & & & & & \searrow & \nearrow & \\ & & & & & & & f_\alpha & \end{array}$$

$\{\psi_{\alpha,\beta}\}$  reparametrization of  $\varphi_\alpha$   
 $\{\xi_{\alpha,\beta,\gamma}\}$  reparametrization of  $f_\alpha \circ \psi_{\alpha,\beta}$   $\Rightarrow$   $\{\varphi_\alpha \circ \psi_{\alpha,\beta} \circ \xi_{\alpha,\beta,\gamma}\}$  reparametrization of  $f$

□

Next we reduce to functions that are  $C^k$ -smooth everywhere, not only in the last  $m$  variables.

**Reduction Step 3.** *Reduction to  $f : I^{m+1} \rightarrow \mathbb{R}$   $k$ -bounded in the last  $m$  variables and  $C^k$ -smooth.*

*Proof.* By d-minimal generic smoothness,  $f$  is  $C^k$ -smooth outside a set  $V \subset I^{m+1}$  of dimension  $\leq m$ . We may reparametrize  $f$  separately on  $V$  and  $I^{m+1} \setminus V$ .

First, we look at  $f|_V$ : By (I) $_{m+1}$ , there is a countable cellular  $k$ -parametrization  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow V\}$  of  $V$ . By Remark 3.3.1 it suffices to find countable cellular reparametrizations for each  $f \circ \varphi_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R}$ . But  $\mathcal{C}_\alpha$  must have some  $\{0\}$ -coordinate as  $V$  has dimension  $\leq m$  and cellular maps preserve dimension. Hence one can reparametrize using (II) $_m$ .

Now we aim to reparametrize  $f|_{I^{m+1} \setminus V}$ : Again, use (I) $_{m+1}$  to obtain a countable cellular  $k$ -parametrization  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow I^{m+1} \setminus V\}$  of  $I^{m+1} \setminus V$ , it suffices to find a reparametrization for each  $f \circ \varphi_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R}$ . As now argued several times above, we may assume  $\mathcal{C}_\alpha = I^{m+1}$ , otherwise we are done by (II) $_m$ . Now note that  $f \circ \varphi_\alpha : I^{m+1} \rightarrow \mathbb{R}$  is  $C^k$ -smooth by choice of  $V$ . Moreover,  $f \circ \varphi_\alpha$  is still  $k$ -bounded in the last  $m$  variables by Lemma 3.3.7.

Replacing  $f$  by  $f \circ \varphi_\alpha$ , we may from now on assume that  $f : I^{m+1} \rightarrow \mathbb{R}$  is  $C^k$ -smooth and  $f^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^m$ ,  $|\mu| \leq k$ ,  $\mu_1 = 0$ .  $\square$

What is left to do is to find an adequate reparametrization of the  $x_1$  variable in order to get the desired bounds also for the derivatives including the  $x_1$  direction. To this end, we use the following crucial lemma that connects the boundedness of derivatives in the last variables to boundedness of the derivative in the  $x_1$  direction.

**Lemma 3.3.8.** *Let  $l \in \mathbb{N}$ ,  $f : I^l \rightarrow I$  a definable  $C^1$ -function and suppose that  $\frac{\partial f}{\partial x_j}$  is bounded for all  $j = 2, \dots, l$ . Then the function  $\frac{\partial f}{\partial x_1}(x_1, -)$  is bounded for all but countably many fixed  $x_1 \in I$ .*

This is a d-minimal version of [2, Lemma 12], which in turn is the second major simplification Binyamini and Novikov made to Pila and Wilkie's original paper (apart from using cellular parametrizations). Pila and Wilkie couldn't prove the boundedness of the derivatives in question a priori, but instead approximate  $f$  by a family of functions with smaller domain for which the claim holds. For each of these they find a reparametrization as needed and prove that these reparametrizations must converge to a reparametrization of the original function which has the needed properties [33, §4].

The proof in [2] relies on uniform finiteness, a property not available in the d-minimal setting. We postpone our proof to Section 3.4.

Using Lemma 3.3.8, we can finish the proof of the second induction step and thus of the countable parametrization theorem.

*Proof of (I) $_{m+1}$  + (II) $_m \Rightarrow$  (II) $_{m+1}$ .* Recall the situation: We are given a definable  $C^k$ -map  $f : I^{m+1} \rightarrow \mathbb{R}$  s.t.  $f^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^{m+1}$  with  $|\mu| \leq k$ ,  $\mu_1 = 0$ . We have to show that  $f$  admits a countable cellular  $k$ -reparametrization.

Let  $\nu \in \mathbb{N}^{m+1}$  be the smallest index in lexicographic order s.t.  $|\nu| \leq k$  and  $f^{(\nu)}$  unbounded. If no such  $\nu$  exists we are already done, as then the identity on  $I^{m+1}$  gives a countable cellular reparametrization of  $f$ . We will find a countable cellular  $k$ -parametrization  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow I\}$  of  $I$  s.t.  $f_\alpha := f \circ (\varphi_\alpha, \text{id}_{I^m})$  has strictly larger  $\nu$  for all  $\alpha$  with  $\mathcal{C}_\alpha = I$ . The claim then follows by induction on  $\nu$ , using that  $f_\alpha$  satisfies the same assumptions we put on  $f$  (or, if  $\mathcal{C}_\alpha = \{0\}$ , can be reparametrized using (II) $_m$ ).

By our assumption on  $f$ , we know that  $\nu_1 > 0$ . Set  $\nu' = \nu - (1, 0, \dots, 0)$ . Then  $\frac{\partial f^{(\nu')}}{\partial x_j}$  is bounded for all  $j = 2, \dots, l+1$  as all the occurring degrees are smaller than  $\nu$  in lexicographic order. Hence we can apply Lemma 3.3.8 to  $f^{(\nu')}$ . Let  $\Sigma \subset I$  denote the countable set of  $x_1$  for which  $\frac{\partial f^{(\nu')}}{\partial x_1}(x_1, -) = f^{(\nu)}(x_1, -)$  is unbounded. On each of the countably many connected components of  $\Sigma \times I^m$  we can reparametrize  $f$  using  $(\text{II})_m$  (as they are  $m$ -dimensional). Considering each of the connected components of  $(I \setminus \Sigma) \times I^m$  separately and rescaling to  $I^{m+1}$  we may assume that  $f^{(\nu)}(x_1, -)$  is bounded for all  $x_1 \in I$ . Now set

$$S := \{x = (x_1, \dots, x_{m+1}) \in I^{m+1} \mid |f^{(\nu)}(x)| \geq \frac{1}{2} \sup_{y \in I^m} |f^{(\nu)}(x_1, y)|\}.$$

This is definable. By definition of  $S$ , for any fixed  $x_1 \in I$  there must exist a  $y \in I^m$  s.t.  $(x_1, y) \in S$ . Using definable choice, we even get a definable curve  $\gamma : I \rightarrow S$  s.t.  $\gamma_1 = \text{id}_I$  where  $\gamma_1, \dots, \gamma_{m+1}$  denote the coordinates of  $\gamma$ . By d-minimal generic smoothness,  $\gamma$  is  $C^1$ -smooth outside a countable set  $\Sigma$ . Applying  $(\text{II})_m$  to  $\Sigma \times I^m$  as above and rescaling, we may assume wlog that  $\gamma$  is  $C^1$ -smooth. By  $(\text{II})_1$  we can find a countable cellular  $k$ -reparametrization  $\{\varphi_\alpha : \mathcal{C}_\alpha \rightarrow I\}$  of  $(\gamma, f^{(\nu')} \circ \gamma)$ .

Fix an  $\alpha$  with  $\mathcal{C}_\alpha = I$ . Set  $\varphi := \varphi_\alpha$ . As outlined above, the only thing left to prove is that  $f_\alpha = f \circ (\varphi, \text{id}_{I^m})$  has smaller  $\nu$  than  $f$ . That means, we have to show that  $f_\alpha^{(\mu)}$  is bounded for all  $\mu \in \mathbb{N}^{m+1}$  with  $|\mu| \leq k$  and  $\mu \leq \nu$  in the lexicographic order. Let such a  $\mu$  be given.

For  $\mu < \nu$ , we know that all the  $f^{(\mu)}$  are bounded as well as the derivatives of  $(\varphi, \text{id}_{I^m})$  up to order  $k$ . The chain rule immediately implies that  $f_\alpha^{(\mu)}$  is bounded.

The only thing left to show is that  $f_\alpha^{(\nu)}$  is bounded. Computing  $f_\alpha^{(\nu)}(x_1, \dots, x_{m+1})$  using the chain rule, the only summand that is potentially unbounded is

$$f^{(\nu)}(\varphi(x_1), x_2, \dots, x_{m+1}) \cdot \varphi'(x_1)^{\nu_1}.$$

Now for all  $(x_1, \dots, x_{m+1}) \in I^{m+1}$

$$\begin{aligned} f^{(\nu)}(\varphi(x_1), x_2, \dots, x_{m+1}) &\leq \sup_{y \in I^m} |f^{(\nu)}(\varphi(x_1), y)| \\ &\leq 2 \cdot |f^{(\nu)}(\gamma(\varphi(x_1)))| \end{aligned}$$

by choice of  $\gamma$ . So it suffices to show that  $(f^{(\nu)} \circ \gamma \circ \varphi) \cdot (\varphi')^{\nu_1} : I \rightarrow I$  is bounded. As  $\varphi'$  is bounded and  $\nu_1 > 0$ , it even suffices to show that  $(f^{(\nu)} \circ \gamma \circ \varphi) \cdot \varphi' : I \rightarrow I$  is bounded. To show this, we compute (again with the chain rule):

$$(f^{(\nu)} \circ \gamma \circ \varphi)' = \left( \left( f^{(\nu)} \circ \gamma + \sum_{i=2}^{m+1} \left( \frac{\partial f^{(\nu')}}{\partial x_i} \circ \gamma \right) \cdot \gamma'_i \right) \circ \varphi \right) \cdot \varphi'$$

The left hand side and the terms arising from the big sum are bounded by choice of  $\varphi$ . Hence also  $(f^{(\nu)} \circ \gamma \circ \varphi) \cdot \varphi'$  must be bounded.  $\square$

### 3.4 Proof of Lemma 3.3.8

For the proof of Lemma 3.3.8 we cannot simply adapt Binyamini and Novikov's proof of the o-minimal version as this crucially uses uniform finiteness.



Instead, we start with the following lemma that is concerned with the uniform boundedness of derivatives in definable families of functions with *discrete* parameter space. The key observation used in the proof is the same that was used by Binyamini and Novikov [2, Lemma 11]: The derivative of a function with range  $I$  may not be greater than  $C \in \mathbb{R}$  on an interval of length  $\frac{1}{C}$ ; otherwise the function would leave  $I$ .

**Lemma 3.4.1.** *Let  $\{f_M : I \rightarrow I\}_{M \in D}$  be a definable family of  $C^1$ -functions with  $D \subset \mathbb{R}$  discrete, closed and infinite ( $\Leftrightarrow$  infinite without limit point). Then there exists an open interval  $J \subset I$  s.t.*

$$\{(f_M)'(s) \mid s \in J, M \in D\}$$

*is bounded.*

*Proof.* Define

$$A := \{s \in I \mid \text{the function } M \mapsto f'_M(s) \text{ is unbounded}\}$$

Certainly,  $A$  is definable. Assume we have shown  $\dim A = 0$ . Then consider the definable function  $g : I \setminus A \rightarrow \mathbb{R}$  given by

$$g(s) = \sup\{|f'_M(s)| \mid M \in D\}.$$

As  $\dim(I \setminus A) = 1$ , by d-minimal generic smoothness there is an interval on which  $g$  is continuous and thus, shrinking further, also an interval  $J$  on which  $g$  is bounded. By definition of  $g$ , this implies the claim.

Left to show is  $\dim A = 0$ . Suppose not. By the family version of the d-minimal monotonicity theorem, there is  $\{X_M\}_{M \in D}$ , a definable family of countable subsets of  $A$ , s.t.  $f'_M$  is monotone on each connected component of  $A \setminus X_M$ . Then also the definable set  $X := \bigcup_{M \in D} X_M$  is countable, so its complement  $A \setminus X$  contains an interval  $(a, b) \subset A$  on which then all the  $f'_M$  must be monotone for  $M \in D$ .

Let  $s_0 = \frac{a+b}{2}$  be its midpoint. As  $s_0 \in A$ , there exists  $M \in D$  with  $|f'_M(s_0)| > \frac{2}{b-a}$ . But then, as  $f'_M$  is monotone on  $(a, b)$ , there is an interval of length  $\frac{b-a}{2}$  on which  $|f'_M|$  is greater than  $\frac{2}{b-a}$  (either the left or the right half of  $(a, b)$ ). This is a contradiction, as  $f_M$  would have to leave  $I$  on this interval.  $\square$

**Remark 3.4.2.** If  $\mathcal{R}$  is d-minimal, but not o-minimal, a definable set  $D \subset \mathbb{R}_+$  which is discrete, closed and infinite, must always exist: As  $\mathcal{R}$  is not o-minimal, there is a definable set  $X \subset \mathbb{R}_+$  which is countably infinite. If  $X$  has no limit point, we are done. If  $X$  has a limit point we can apply the definable function given by taking the distance to  $\text{cl}(\text{lim}(X))$  and get a definable subset of  $\mathbb{R}_+$  with unique limit point 0. Cutting out 0, applying the multiplicative inverse function and cutting out  $(0, 1)$ , we get a definable  $D \subset \mathbb{R}_+$  as desired.

We can generalize Lemma 3.4.1 to finitely many families at once, making use of the structure of the parameter space  $D$ :

**Corollary 3.4.3.** *Let  $l \in \mathbb{N}$ . For  $i = 1, \dots, l$  let  $\{f_M^i : I \rightarrow I\}_{M \in D}$  be a definable family of  $C^1$ -functions with  $D \subset \mathbb{R}_+$  discrete, closed and infinite. Then there exists an open interval  $J \subset I$  s.t.*

$$\{(f_M^i)'(s) \mid s \in J, M \in D, i = 1, \dots, l\}$$

*is bounded.*



*Proof.* We construct a new index set  $D' \subset \mathbb{R}$  discrete, closed, infinite and a new definable family  $g = \{g_K : I \rightarrow I\}_{K \in D'}$  of  $C^1$ -functions s.t. for all  $i \in \{1, \dots, l\}$  and  $M \in D$  there exists  $K \in D'$  with  $g_K = f_M^i$ . Then applying Lemma 3.4.1 to  $g$  immediately implies the claim.

For  $M \in D$  let  $\sigma(M) \in D$  be its successor in  $D$ , that is  $\min(D \setminus (-\infty, M])$  (well-defined by the assumptions on  $D$ ). Note that  $\sigma : D \rightarrow D$  is definable. Now define

$$D' := \left\{ M, M + \frac{r}{l}, M + \frac{2r}{l}, \dots, M + \frac{(l-1)r}{l} \mid M \in D, r = \sigma(M) - M \right\}$$

Clearly  $D'$  is again discrete, closed, infinite and definable. Set  $g_K := f_M^i$  for  $K \in D'$  of the form  $K = M + \frac{(i-1)(\sigma(M)-M)}{l}$ . Then the family  $g = \{g_K : I \rightarrow I\}_{K \in D'}$  is definable and  $g$  and  $D'$  have the desired properties.  $\square$

Now we can proof Lemma 3.3.8. Recall the statement:

**Lemma 3.3.8.** *Let  $l \in \mathbb{N}$ ,  $f : I^l \rightarrow I$  a definable  $C^1$ -function and suppose that  $\frac{\partial f}{\partial x_j}$  is bounded for all  $j = 2, \dots, l$ . Then the function  $\frac{\partial f}{\partial x_1}(x_1, -)$  is bounded for all but countably many fixed  $x_1 \in I$ .*

*Proof.* If our structure  $\mathcal{R}$  is o-minimal, this is [2, Lemma 12]. Hence we may assume that  $\mathcal{R}$  is d-minimal, but not o-minimal. Then there exists  $D \subset \mathbb{R}_+$  definable, discrete, closed and infinite (cf. Remark 3.4.2).

For a contrary, assume that the set  $\{x_1 \in I \mid \frac{\partial f}{\partial x_1}(x_1, -) \text{ is unbounded}\}$  is uncountable. Then, by d-minimality, it must contain an interval. After restriction and rescaling, we may assume wlog that it is the whole  $I$ .

By definable choice, there is a definable family of curves  $\{\gamma_M : I \rightarrow I^{l-1}\}_{M \in D}$  such that for all  $M \in D$ ,  $s \in I$ :

$$\left| \frac{\partial f}{\partial x_1}(s, \gamma_M(s)) \right| > M \quad (3.2)$$

By the family version of d-minimal generic smoothness, we get  $\{X_M\}_{M \in D}$ , a definable family of countable subsets of  $I$ , s.t.  $\gamma_M$  is  $C^1$ -smooth on  $I \setminus X_M$ . Then also the definable set  $X := \bigcup_{M \in D} X_M$  is countable, so its complement  $I \setminus X$  contains an interval on which then all the  $\gamma_M$  need to be  $C^1$ -smooth. After restriction and rescaling, we may assume wlog that this interval is the whole  $I$ , i.e. that the  $\gamma_M$  are  $C^1$ -smooth.

Denote the coordinates of  $\gamma_M$  by  $\gamma_M^1, \dots, \gamma_M^{l-1}$ . Apply Corollary 3.4.3 to the  $\{\gamma_M^i : I \rightarrow I^{l-1}\}_{M \in D}$  for  $i = 1, \dots, l-1$  and to  $\{f(-, \gamma_M(-)) : I \rightarrow I\}_{M \in D}$ . This gives an interval  $J \subset I$  and a constant bound  $C_1 \in \mathbb{R}_+$  s.t. for all  $s \in J$  and  $M \in D$  we have:

$$|(\gamma_M^i)'(s)| \leq C_1 \quad \text{for } i = 1, \dots, l-1, \text{ and} \quad (3.3)$$

$$|f(-, \gamma_M(-))'(s)| \leq C_1 \quad (3.4)$$

On the other hand, by the chain rule, for  $s \in J$ ,  $M \in D$ :

$$\begin{aligned} |f(-, \gamma_M(-))'(s)| &= \left| \frac{\partial f}{\partial x_1}(s, \gamma_M(s)) + \sum_{i=2}^l \frac{\partial f}{\partial x_i}(s, \gamma_M(s)) \cdot (\gamma_M^{i-1})'(s) \right| \\ &\geq \left| \frac{\partial f}{\partial x_1}(s, \gamma_M(s)) \right| - \sum_{i=2}^l \left| \frac{\partial f}{\partial x_i}(s, \gamma_M(s)) \right| \cdot |(\gamma_M^{i-1})'(s)| \\ &\stackrel{(3.2)+(3.3)}{>} M - (l-1) \cdot C_1 \cdot C_2, \end{aligned}$$

where  $C_2 \in \mathbb{R}_+$  is an upper bound for all  $|\frac{\partial f}{\partial x_j}|$  for  $j = 2, \dots, l$  which exists by assumption. Now choosing  $M \in D$  with  $M > (l-1) \cdot C_1 \cdot C_2 + C_1$  gives a contradiction to (3.4) (use that  $D$  is unbounded).  $\square$

### 3.5 Family versions of the countable parametrization theorem

In the proof of the second induction step we needed a family version of the reparametrization theorem  $(\text{II})_m$ . In the o-minimal case, this can be deduced from  $(\text{II})_m$  by using a model-theoretic argument involving definable choice and passing over to an elementary extension [2, §3.5].

This is not possible in the d-minimal case as one would need an infinite first-order sentence to refer to all the functions of a parametrization in one formula, at least with the current conception of countable parametrizations.

However, one can instead obtain the family version by adding parameters to all statements in the proof in Section 3.3. Conceding that we have actually been proving the family versions from the beginning and have just simplified notation, we are allowed to use the family version of  $(\text{II})_m$  in the proof of the second induction step.

In this section, we state the needed family versions of parametrization and reparametrization, clarify how to show the family version of  $(\text{I})_1$  and comment on the adaptations to the proof in Section 3.3 that have to be made in order to prove the family versions. The adequate family versions of  $(\text{I})_m$  and  $(\text{II})_m$  are:

**Theorem 3.5.1.** *Let  $m, l \in \mathbb{N}$ . Let  $\{X_s\}_{s \in I^l}$  be a definable family of uniformly bounded subsets of  $\mathbb{R}^m$ . Then for all  $k \in \mathbb{N}$ , there exist countably many definable families of functions  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow X_s\}_{s \in I^l}$  and basic cells  $\mathcal{C}_\alpha$  s.t.*

- (i) *For all  $\alpha$  and  $s \in I^l$ , either  $\mathcal{C}_{\alpha, s} = \mathcal{C}_\alpha$  or  $\mathcal{C}_{\alpha, s} = \emptyset$ .*
- (ii) *For fixed  $s \in I^l$ ,  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow X_s\}$  forms a  $k$ -parametrization of  $X_s$  (ignoring the possibly empty functions).*
- (iii) *For fixed  $\alpha$ , the derivatives of  $\varphi_{\alpha, s}$  up to order  $k$  are even bounded uniformly in  $s$ .*

**Theorem 3.5.2.** *Let  $m, n, l \in \mathbb{N}$ . Let  $\{f_s : X_s \rightarrow \mathbb{R}^n\}_{s \in I^l}$  be a definable family of uniformly bounded functions with  $X_s \subset \mathbb{R}^m$ . Then for all  $k \in \mathbb{N}$ , there exist countably many definable families of functions  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow X_s\}_{s \in I^l}$  and basic cells  $\mathcal{C}_\alpha$  s.t.*

- (i) *For all  $\alpha$  and  $s \in I^l$ , either  $\mathcal{C}_{\alpha, s} = \mathcal{C}_\alpha$  or  $\mathcal{C}_{\alpha, s} = \emptyset$ .*
- (ii) *For fixed  $s \in I^l$ ,  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow X_s\}$  forms a  $k$ -reparametrization of  $f_s$  (ignoring the possibly empty functions).*
- (iii) *For fixed  $\alpha$ , the derivatives of  $\varphi_{\alpha, s}$  and  $f_s \circ \varphi_{\alpha, s}$  up to order  $k$  are even bounded uniformly in  $s$ .*

The base case  $m = 1$  in the family version of parametrization holds by countable cell decomposition:

*Proof of the case  $m = 1$  in Theorem 3.5.1.* Denote  $X := \{X_s\}_{s \in I^l} \subset I^l \times \mathbb{R}$ . By countable cell decomposition,  $X$  is the union of countably many cells  $A^\alpha$ . Fix one  $\alpha$ . If  $A^\alpha$  is of the form  $(f, g)_Y$  for some cell  $Y \subset I^l$  and  $f, g : Y \rightarrow \mathbb{R}$  bounded continuous definable

functions with  $f < g$ , any fiber  $A_s^\alpha$  for  $s \in I^l$  is either empty or an interval by the structure of  $(f, g)_Y$ . Then we can parametrize the  $A_s^\alpha$  uniformly with a family of linear strictly increasing bijections  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow A_s^\alpha\}_{s \in I^l}$  with  $\mathcal{C}_{\alpha, s}$  either  $I$  or empty, depending on  $A_s^\alpha$  being an interval or empty.

If on the other hand  $A^\alpha$  is the graph of a function, then  $A_s^\alpha$  is either empty or a point for all  $s \in I^l$  and we can similarly parametrize the  $A_s^\alpha$  uniformly with a definable family of maps  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow A_s^\alpha\}_{s \in I^l}$  (now  $\mathcal{C}_{\alpha, s}$  either empty or  $\{0\}$ ).

As the  $A_s^\alpha$  are bounded uniformly in  $s$ , we also get uniform bounds for the derivatives of the  $\varphi_{\alpha, s}$ . Then for fixed  $s \in I^l$ ,  $\{\varphi_{\alpha, s} : \mathcal{C}_{\alpha, s} \rightarrow X_s\}$  forms a  $k$ -parametrization of  $X_s$  (as  $X_s = \bigcup_\alpha A_s^\alpha$ ) and also the other conditions are satisfied by construction.  $\square$

Now Theorems 3.5.1 and 3.5.2 can be proven essentially the same way as the absolute versions in the preceding Section 3.3. The following adaptations have to be made:

- One has to add parameters to all statements (i.e. turn any definable set into a definable family of sets, and any definable function into a definable family of functions, parametrized by  $s \in I^l$ ).
- Any application of countable cell decomposition, monotonicity theorem or generic smoothness has to be replaced by an application of the corresponding family version (see Chapter 2).
- At any occasion where something is bounded, it has to be bounded uniformly in  $s$ .
- At any time where the domain of a function is explicitly assumed to be some set  $A$ , one has to assume instead that the fibers of the domain are either  $A$  or empty.
- At any occasion where an interval or a point is singled out from a definable subset of  $\mathbb{R}$ , countable cell decomposition has to be used to do this uniformly, exactly the same way as in the above proof of the case  $m = 1$  of Theorem 3.5.1.

### 3.6 Towards a d-minimal Pila-Wilkie theorem

Unfortunately, the countable parametrization theorem is not strong enough to directly deduce a version of the Pila-Wilkie theorem in d-minimal structures. By Pila's theorem (Theorem 3.1.5), the rational points (of bounded height) in the image of one partial parametrization lie in few hypersurfaces. If one can cover a definable set  $X$  with finitely many partial parametrizations, as in the original parametrization theorem, this property carries over to the whole set  $X$ . Then one can reduce to counting rational points in  $X \cap H$  for  $H$  a hypersurface of suitable degree. However, with infinitely many partial parametrizations, a priori the rational points instead lie in infinitely many hypersurfaces and the reduction does not work.

To prove a d-minimal Pila-Wilkie theorem, we need more information on the parametrization beyond just being 'countable'. In particular, it would be desirable to get a bound on the number of partial parametrizations that are needed to cover the rational points of a given height:

**Conjecture.** *Let  $X \subset \mathbb{R}^n$  definable,  $k \in \mathbb{N}$  and  $\varepsilon > 0$ . Then there is a constant  $C \in \mathbb{R}$  s.t. for all  $T \geq 1$ ,  $X(\mathbb{Q}, T)$  can be covered by the images of at most  $C \cdot T^\varepsilon$  partial  $k$ -parametrizations of  $X$ .*

A possible technique to get from infinitely to finitely many partial parametrizations is to approximate  $X$  uniformly by subsets which admit a finite parametrization. For example, if  $\{\varphi_i\}_{i \in \mathbb{N}}$  is a countable parametrization of  $X$ , we can write  $X$  as

$$X = \bigcup_{i \in \mathbb{N}} \underbrace{\bigcup_{j \leq i} \text{im}(\varphi_j)}_{=: Y_i}$$

and the  $Y_i$  form an increasing sequence of sets converging to  $X$ , each  $Y_i$  admitting a finite parametrization. If the  $Y_i$  could be defined uniformly, that is, using one formula for all  $Y_i$ , then assertions about the  $Y_i$  may potentially be transferred to  $X$ .

Such a uniform approximation could also be a key to an improved d-minimal cell decomposition theorem with more control about the position or size of the cells (cf. the discussion in Section 2.3). Cell decomposition and parametrization seem to be ultimately connected and one can hope that a better d-minimal cell decomposition theorem also yields an adequate parametrization theorem.

Therefore, one has to analyze how the decomposition into cells and the decomposition into images of partial parametrizations interact: Do we have control about the number of partial parametrizations needed to cover a cell? A cell does not necessarily admit a finite parametrization, for example, if it is the graph of a strongly oscillating function. However, it may possibly be approximated in a uniform way by even simpler sets, in the same style as above. Does, for example, the graph of a monotone, smooth function with 'nice' derivatives allow for a good bound on the number of partial parametrizations needed?

This thesis has to stop, while research never ends.

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