The Pila-Zannier method for Abelian varieties

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Introduction

The present thesis concerns a quite recent solution to a classical problem: the *Pila-Zannier proof* of the *Manin-Mumford conjecture*. In 2008, Pila and Zannier gave a substantially new proof to a Diophantine problem using mathematical logic, more precisely using o-minimal geometry [PZ08]. Following their method, we present a proof of the Manin-Mumford conjecture in the following form:

Theorem. *Let A be a complex Abelian variety and let $V \subseteq A$ be a closed algebraic subvariety. Suppose that V does not contain any translate of a positive dimensional Abelian subvariety of A. Then V contains only finitely many torsion points of A.*

This statement was first proved by Raynaud in 1983 [Ray83]. The Pila-Zannier proof differs from Raynaud's proof completely and is remarkable for several reasons. It is the first proof of Manin-Mumford relying on real methods. Moreover, the method of the proof has turned out to be applicable to problems going far beyond the case of Abelian varieties. As recently as 2021, Pila, Shankar and Tsimerman proved the *André-Oort conjecture* using the Pila-Zannier strategy, thereby giving the first proof of André-Oort in full generality [PST21].

The Pila-Zannier method exposes a connection between o-minimal geometry and Diphantine geometry. These are two subjects which we, at first glance, would not expect to harmonise well. O-minimal geometry is a branch of model theory concerning structures with a dense linear order and an extra condition that yields a tame geometric structure theory for the definable sets. Diophantine geometry belongs to algebraic geometry and studies the geometry and arithmetic of solution sets to polynomial equations, often aiming to understand *special points* on varieties such as torsion points with respect to a group structure. Due to the presence of order, o-minimality seems to be a framework suitable for applications in analysis, rather than algebra. In particular, interesting examples of o-minimal structures arise as expansions of the real ordered field. Contrary to this, in Diophantine geometry, we classically work over an algebraically closed field, which cannot be an ordered field. Definable sets in expansions of the real ordered field are naturally equipped with the Euclidean topology, whereas varieties are usually equipped the Zariski topology. The success of the Pila-Zannier method is thus, at least in view of this naive comparison, very surprising.

We begin with looking at a problem related to the Manin-Mumford conjecture, known as *Mann's Theorem*. This will provide a motivation for the key ideas of the Pila-Zannier method without requiring much abstract machinery (Section 1). Afterwards, we introduce and discuss *Abelian varieties*, providing the relevant background from algebraic geometry (Section 2). We then come to the model theoretic foundation of the Pila-Zannier method, most importantly to the *Pila-Wilkie Counting Theorem* (Section 3). Based on these foundations, we will be able to obtain a proof of *Manin-Mumford* (Section 4).

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Contents

Conventions

The *natural numbers* $\mathbb{N}$ contain 0. The *cardinality* of a set M is denoted by $\#M$. For an ordered set M and $a \in M$, we write $M_{\geq a}$ for $\{x \in M : x \geq a\}$, and similarly $M_{>a}$.

Let R be a ring and $m \in \mathbb{N}_{\geq 1}$. We donte elements in the *polynomial ring* $R[X_1, \ldots, X_m]$ or in the *ring of formal power series* $R[\![X_1, \ldots, X_m]\!]$ in m variables over R by $\sum_{\alpha \in \mathbb{N}^m} b_\alpha X^\alpha$, where $b_\alpha \in R$ and $X^\alpha := X_1^{\alpha_1} \cdots X_m^{\alpha_m}$. For $\alpha \in \mathbb{N}^m$, we set $|\alpha| = \sum_{i=1}^m \alpha_i$.

Let k be a field and $m \in \mathbb{N}_{\geq 1}$. An *algebraic set* in k^m is the zero set of polynomials $f_1, \ldots, f_r$ in $k[X_1, \ldots, X_m]$ for some $r \in \mathbb{N}_{\geq 1}$. We will then denote it by $V(f_1, \ldots, f_r)$.

The term *definable* will always mean *definable with parameters*. Structures $\mathcal{M}, \mathcal{N}, \ldots$ are understood to have the underlying sets $M, N, \ldots$, respectively. We fix the *language of ordered rings* $\mathcal{L}_{\mathrm{oring}} = \{+, -, \cdot, 0, 1, <\}$ and fix the *real ordered field* as an $\mathcal{L}_{\mathrm{oring}}$-structure $\mathbb{R}_{\mathrm{alg}}$.

If not specified otherwise, by a *space* we always mean a *topological space*. To avoid confusion, we will sometimes write $\mathrm{cl}_X^{\mathcal{T}}(Y)$ for the closure of Y in X with respect to the topology $\mathcal{T}$, and moreover $\mathrm{cl}^{\mathcal{T}}(Y)$ if the ambient space is fixed.

If G is a group and $X \subseteq G$ a subset, then by a *translate* of X we will always mean a subset of G of the form $aX = \{ax : x \in X\}$ for some $a \in G$. If X is a subgroup of G, this is the usual notion of a *coset*. The set of *torsion elements* of a group G will be denoted by G_{tor}. When we say that a torsion element has *order* n, then we always that it has *exact order* n.

Let k'/k be a field extension. Its *degree* is denoted by $[k' : k]$ and its automorphism group is denoted by $\mathrm{Aut}(k'/k)$. If k'/k is Galois, we also write $\mathrm{Gal}(k'/k)$ for $\mathrm{Aut}(k'/k)$. For a field k, we denote by $\overline{k}$ an algebraic closure of k.

Text in **bold** font indicates definitions, while *italics* emphasise terms for various reasons.

1 Roots of unity on an algebraic set

The aim of this first section is to give an outline of the Pila-Zannier strategy and a motivation for the content of the following sections. We do this hands-on by discussing a problem related to the Manin-Mumford conjecture. The following theorem concerns $(\mathbb{C}^\times)^n$, the n-th cartesian power of the multiplicative group of units in $\mathbb{C}$, which we denote by $\mathbb{G}^n$.

Theorem 1.1. *Let $Y \subseteq \mathbb{C}^n$ be an algebraic set. Suppose that Y does not contain any translate of an infinite algebraic subgroup of $\mathbb{G}^n$. Then the set $Y \cap \mathbb{G}^n_{\mathrm{tor}}$ is finite, i.e., there are only finitely many points $(\zeta_1, \ldots, \zeta_n)$ on Y such that each ζ_j is a root of unity.*

This theorem is often referred to as *multiplicative Manin-Mumford* or *Mann's Theorem* and is easier to prove than Manin-Mumford. In fact, it was shown by Mann in 1965, even before the Manin-Mumford conjecture was formulated [Man65]. Although the resemblance of this statement to Manin-Mumford as stated in the introduction is striking, it is not, strictly speaking, a special case. This is so because $\mathbb{G}^n$ is not an Abelian variety, as will become clear in the section on Abelian varieties (Section 2). We could weaken the statement by replacing *infinite algebraic subgroup* by *infinite subgroup*, thereby omitting the reference to algebraic groups, which will be formally defined in the next section. In this motivating section, by an algebraic subgroup of $\mathbb{G}^n$ me mean an subgroup H of $\mathbb{G}^n$ such that $H = \mathbb{G}^n \cap Z$ for some algebraic set $Z \subseteq \mathbb{C}^n$. We should remark, however, that the statement above is already a weak version. In fact, for any algebraic set $Y \subseteq \mathbb{C}^n$, the intersection $Y \cap \mathbb{G}^n_{\mathrm{tor}}$ is a finite union of translates of algebraic subgroups of $\mathbb{G}^n$ (cf. [Mar10], Section 8).

The geometry is easier in the case of Mann's Theorem than in the case of Manin-Mumford, but the three key ingredients *o-minimality*, *Galois bounds* and *functional transcendence* play the same role in both proofs. The *Pila-Zannier method* proceeds in the following steps:

1. Transfer the problem to an o-minimal structure. By doing so, reduce the problem to the study of the *algebraic* and the *transcendental* part of a definable set.

2. Prove that the transcendental part in question yields finitely many torsion points by playing upper bounds from o-minimality against lower bounds from Galois theory.

3. Show that the algebraic part in question yields a well-structured set of torsion points by proving or applying functional transcendence results.

These general three steps will be elaborated for Mann's Theorem in the following three subsections. Since this section has mainly a motivating purpose, we delay some details until later sections. Our presentation is based on two survey articles by Scanlon [Sca12, Sca17].

1.1 O-minimality enters the stage

As outlined in the introduction, o-minimality is known to be a framework suitable for applications in an analytic rather than an algebraic setting. Mann's Theorem is stated purely in algebraic terms, so let us first *analytify* the problem. There is a complex analytic group homomorphism

$$E\colon \mathbb{C}^n \to \mathbb{G}^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}).$$

For $a \in \mathbb{C}$, the nonzero complex number $e^{2\pi i a}$ is a root of unity if and only if $a \in \mathbb{Q}$. Hence, $\zeta \in \mathbb{G}^n$ is a tuple of roots of unity if and only if there is $z \in \mathbb{Q}^n$ such that $E(z) = \zeta$.

Let $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_n]$ be polynomials whose zero set is Y. In order to understand roots of unity on Y, we could try to understand rational solutions to the transcendental equations $f_k(E(z)) = 0$ for all $k \in \{1, \ldots, r\}$, i.e. the rational points of the set

$$E^{-1}(Y) = \{z \in \mathbb{C}^n : f_k(E(z)) = 0 \text{ for all } k \in \{1, \ldots, r\}\}.$$

This is an *analytic* set in $\mathbb{C}^n$ (for a definition, see (4.4)). At first glance, it may seem like we have converted a difficult problem into an unsolvable problem, since understanding rational points on analytic sets is notoriously hard. Even if we assume that our analytic set is in fact algebraic, understanding its rational points is (arguably) the fundamental problem of arithmetic geometry. Also from a model theoretic point of view, we cannot expect much in this situation. It is natural to consider $E^{-1}(Y)$ as a definable set in the structure $\mathbb{C}_{\exp} = (\mathbb{C}, +, -, \cdot, 0, 1, \exp)$. Since also $\mathbb{Z} = \{z \in \mathbb{C} : e^{2\pi i z} = 1\}$ is definable in $\mathbb{C}_{\exp}$ and inherits multiplication, the structure $\mathbb{C}_{\exp}$ suffers from Gödel's Incompleteness Theorems and its definable sets are far from being tame. The next lemma provides a solution to this problem.

Lemma 1.2 (Key observation)**.** *The map E restricts to a bijection $[0,1)^n \cap \mathbb{Q}^n \to \mathbb{G}^n_{\mathrm{tor}}$.*

Proof. This follows from basic properties of the complex exponential function. The map $[0,1) \to S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, $z \mapsto e^{2\pi i z}$ is bijective. As $z \in [0,1)$ is rational if and only if $e^{2\pi i z}$ is a root of unity, the map restricts to a bijection $[0,1) \cap \mathbb{Q} \to \mathbb{G}_{\mathrm{tor}}$, which is the desired bijection in each coordinate. □

So we can restrict E to the smaller domain $\mathcal{F} := [0,1)^n$ without losing torsion points in the image. Thus, we define $\tilde{E} \colon \mathcal{F} \to \mathbb{G}^n$ as the restriction $E|_{\mathcal{F}}$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by separating real and imaginary parts, i.e. with the isomorphism of real vector spaces

$$\Phi \colon \mathbb{C}^n \to \mathbb{R}^{2n}, \quad (z_1, \ldots, z_n) \mapsto (\mathrm{Re}(z_1), \mathrm{Im}(z_1), \ldots, \mathrm{Re}(z_n), \mathrm{Im}(z_n)).$$

With this identification, $\tilde{E} \colon [0,1)^n \to \mathbb{R}^{2n}$ is definable in $\mathbb{R}_{\mathrm{an}}$ (for a definition of this structure, see (3.2)). In more detail, the map $\Phi|_{\mathbb{G}^n} \circ \tilde{E}$ is given by

$$[0,1)^n \to \mathbb{R}^{2n}, \quad (x_1, \ldots, x_n) \mapsto (\cos(2\pi x_j), \sin(2\pi x_j))_{j=1}^n,$$

and thus indeed definable in $\mathbb{R}_{\mathrm{an}}$. Again, by separating real and imaginary parts, we see that the polynomials f_j for $j \in \{1, \ldots, r\}$ whose zero set is Y yield functions $\mathbb{R}^{2n} \to \mathbb{R}^2$ that are real polynomials in each coordinate, hence also definable in $\mathbb{R}_{\mathrm{an}}$. We conclude that the set

$$X := \tilde{E}^{-1}(Y) = \{z \in \mathcal{F} : f_k(\tilde{E}(z)) = 0 \text{ for all } k \in \{1, \ldots, r\}\} \subseteq [0,1)^n \subseteq \mathbb{R}^n$$

is definable in $\mathbb{R}_{\mathrm{an}}$. This definable set X will be our central object of study.

As Łojasievicz and Gabrielov proved implicitly and van den Dries proved explicitly, the structure $\mathbb{R}_{\mathrm{an}}$ is o-minimal [Łoj65, Gab68, vdD86]. The upshot of this is that we have translated our problem to a geometrically tame setting. Some features of o-minimal structures justifying the term *tame* will be discussed later on (Section 3). Postponing this more extensive discussion, let us come to the surprising consequence o-minimality has in the present situation. We will argue that there are *few* rational points on the *transcendental part* of X. To make this precise, we introduce some definitions.

Definition 1.3. We define $\mathsf{h} \colon \mathbb{Q} \to \mathbb{N}_{\geq 1}$ by sending a/b with $a \in \mathbb{N}$, $b \in \mathbb{Z} \setminus \{0\}$ coprime (and $b = 1$ for $a = 0$) to $\max\{|a|, |b|\}$. For $n \in \mathbb{N}_{\geq 1}$, we extend h to the **classical height function**

$$\mathsf{H} \colon \mathbb{Q}^n \to \mathbb{N}_{\geq 1}, \quad (q_1, \ldots, q_n) \mapsto \max\{\mathsf{h}(q_1), \ldots, \mathsf{h}(q_n)\}.$$

For $X \subseteq \mathbb{R}^n$ the **set of rational points** on X, denoted by $X(\mathbb{Q})$, is $X \cap \mathbb{Q}^n$. For $t \in \mathbb{R}_{\geq 0}$ we define $X(\mathbb{Q}, t) := \{x \in X(\mathbb{Q}) : \mathsf{H}(x) \leq t\}$ and $\mathsf{N}(X, t) := \#X(\mathbb{Q}, t)$.

A key idea of the Pila-Zannier method is to partition the definable set X into two subsets, one of which we expect to have a well structured set of rational points and the other of which we expect to have few rational points. For the respective definitions of the *algebraic* and the *transcendental* part of X, we first recall the definition if a semialgebraic set.

Definition 1.4. A set $X \subseteq \mathbb{R}^n$ is **semialgebraic** if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : p(x) = 0, q_1(x) > 0, \dots, q_r(x) > 0\}$$

for some $r \in \mathbb{N}$ and some polynomials $p, q_1, \dots, q_r \in \mathbb{R}[X_1, \dots, X_n]$. For a semialgebraic set $X \subseteq \mathbb{R}^n$, a map $f\colon X \to \mathbb{R}^m$ is **semialgebraic** if its graph is a semialgebraic set in $\mathbb{R}^{n+m}$.

It follows from quantifier elimination for $\mathbb{R}_{\mathrm{alg}}$ (also known as the Tarski-Seidenberg theorem) that the semialgebraic sets are precisely the definable sets in $\mathbb{R}_{\mathrm{alg}}$ and that $\mathbb{R}_{\mathrm{alg}}$ is o-minimal.

Definition 1.5. Let $X \subseteq \mathbb{R}^m$. We define the **algebraic part** of X, denoted by X^{alg}, to be the union of all connected infinite semialgebraic subsets of X. We define the **transcendental part** of X, denoted by X^{trans}, to be $X \setminus X^{\mathrm{alg}}$.

There are various equivalent definitions of the algebraic part, due to the tameness of the o-minimal structure $\mathbb{R}_{\mathrm{alg}}$. For example, there is a well behaved notion of dimension for semialgebraic sets (e.g. via cell decomposition, see (3.6)) and we could replace *infinite* by *positive dimensional*. Another very useful fact is that for all but finitely many $x \in X^{\mathrm{alg}}$ we find a continuous semialgebraic nonconstant map $\gamma\colon (-1, 1) \to X$ with x in its image. This can be deduced, for example, from the Cell Decomposition Theorem for o-minimal structures (3.6). Since $\mathbb{R}_{\mathrm{alg}}$ admits an analytic cell decomposition (cf. Remark 3.33 in [Sca17]), we can even assume that γ is analytic (Lemma 3.36 in [Sca17]). The fact that we might miss finitely many points is not relevant for the finiteness result we are aiming for in this section.

In the present situation of Mann's Theorem, we have already introduced a set X, which is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$. Let us extend our key observation (1.2).

Lemma 1.6 (Key observation, extended)**.** *The map E restricts to a bijection*

$$X(\mathbb{Q}) = X \cap \mathbb{Q}^n \to Y \cap \mathbb{G}_{\mathrm{tor}}^n.$$

Moreover, given $x = (a_1/b_1, \dots, a_n/b_n) \in [0, 1)^n \cap \mathbb{Q}^n$ with $a_j \in \mathbb{N}$, $b_j \in \mathbb{N} \setminus \{0\}$ coprime and $b_j = 1$ for $a_j = 0$, then

$$\mathsf{H}(x) = \max\{b_1, \dots, b_n\} \quad \textit{and} \quad \operatorname{ord}(E(x)) = \operatorname{lcm}(b_1, \dots, b_n).$$

In particular, we have $\mathsf{H}(x) \le \operatorname{ord}(E(x)) \le \mathsf{H}(x)^n$.

Proof. The preimage of $Y \cap \mathbb{G}_{\mathrm{tor}}^n$ under the bijection $[0, 1)^n \cap \mathbb{Q}^n \to \mathbb{G}_{\mathrm{tor}}^n$ from the initial key observation (1.2) is precisely $X \cap \mathbb{Q}^n$, so this bijection restricts further to a bijection $X \cap \mathbb{Q}^n \to Y \cap \mathbb{G}_{\mathrm{tor}}^n$. Given $x = (a_1/b_1, \dots, a_n/b_n) \in [0, 1)^n \cap \mathbb{Q}^n$ with $a_j \in \mathbb{N}$, $b_j \in \mathbb{N} \setminus \{0\}$ coprime and $b_j = 1$ for $a_j = 0$, we have $0 \le a_j < b_j$ for all $j \in \{1, \dots, n\}$, so indeed $H(x)$ equals $\max\{b_1, \dots, b_n\}$. On the other hand, as the a_j and b_j are coprime, each $e^{2\pi i a_j/b_j}$ has exact order b_j, so the tuple $E(x) = (e^{2\pi i a_1/b_1}, \dots, e^{2\pi i a_n/b_n}) \in \mathbb{G}_{\mathrm{tor}}^n$ has order $\operatorname{lcm}(b_1, \dots, b_n)$. In particular, we have $H(x) \le \operatorname{ord}(E(x)) \le H(x)^n$ as claimed. □

The upshot of this is that we can bound the *orders* of points on $Y \cap \mathbb{G}_{\mathrm{tor}}^n$ by bounding the *heights* of points on $X(\mathbb{Q})$. Moreover, the set $Y \cap \mathbb{G}_{\mathrm{tor}}^n$ can be written as a disjoint union

$$Y \cap \mathbb{G}_{\mathrm{tor}}^n = E(X^{\mathrm{trans}}(\mathbb{Q})) \cup E(X^{\mathrm{alg}}(\mathbb{Q})).$$

The remaining discussion now naturally splits into two parts. First, we show that $E(X^{\mathrm{trans}}(\mathbb{Q}))$ is finite. The most remarkable ingredient to the proof of this statement is following theorem, which lies at the heart of the Pila-Zannier method:

Theorem 1.7 (Pila-Wilkie Counting Theorem)**.** *Let X be definable in an o-minimal expansion of $\mathbb{R}_{\mathrm{alg}}$. Then for all $\varepsilon > 0$ there exists a constant C, only depending on X and ε, such that $\mathsf{N}(X^{\mathrm{trans}}, t) \le Ct^{\varepsilon}$ for all $t \in \mathbb{R}_{\ge 0}$.*

So roughly speaking, the number of rational points on X^{trans} of height up to t grows slower that any power of t. By this Counting Theorem, there are *few* points in $X^{\text{trans}}(\mathbb{Q})$, and we can conclude that there are *few* points in $E(X^{\text{trans}}(\mathbb{Q}))$. The objective is now to show that there are in fact only *finitely many* torsion points in the image of the transcendental part. This result will be achieved using *Galois theory*.

In a final part, we understand the structure of X^{alg} and show that, under the present assumption of Mann's Theorem, X^{alg} and therefore also $E(X^{\text{alg}}(\mathbb{Q}))$ is empty. This part will use *functional transcendence*.

1.2 From few to finite with Galois theory

In this subsection, we will show that the transcendental part X^{trans} of the definable set X in question is finite. The Counting Theorem (1.7) provides an upper bound for the number of rational points below any given height, but not a finiteness result yet. To get from few to finite, we will play the upper bounds from o-minimality against lower bounds from Galois theory. For this to succeed, we have to work over a number field, which we are allowed to do by the following lemma. Recall that an algebraic set $Y \subseteq \mathbb{C}^n$ is said to be *defined over a subfield k of* $\mathbb{C}$ if Y can be written as the vanishing set of polynomials in $k[X_1, \ldots, X_n]$.

Lemma 1.8. *We may assume that Y is defined over a number field k.*

Proof. Let $Z := \overline{Y \cap \mathbb{G}^n_{\text{tor}}}$ be the Zariski closure of $Y \cap \mathbb{G}^n_{\text{tor}}$ in $\mathbb{C}^n$. We first show that Y and Z contain the same torsion points of $\mathbb{G}^n$. The inclusion $Y \cap \mathbb{G}^n_{\text{tor}} \subseteq Z \cap \mathbb{G}^n_{\text{tor}}$ is immediate. The reverse inclusion holds for purely topological reasons. The set $Y \cap \mathbb{G}^n_{\text{tor}}$ is Zariski closed in $\mathbb{G}^n_{\text{tor}}$, hence $Z \cap \mathbb{G}^n_{\text{tor}} = \overline{Y \cap \mathbb{G}^n_{\text{tor}}} \cap \mathbb{G}^n_{\text{tor}} \subseteq Y \cap \mathbb{G}^n_{\text{tor}}$.

So we can replace Y by Z. All coordinates of points in $\mathbb{G}^n_{\text{tor}}$ are algebraic over $\mathbb{Q}$, being solutions to equations $X^\ell - 1$ for some $\ell \in \mathbb{N}_{\geq 1}$. Hence Y contains the Zariski dense set $Y \cap \mathbb{G}^n_{\text{tor}}$ of points with coordinates in $\overline{\mathbb{Q}}$.

Claim. If an algebraic set $Y \subseteq \mathbb{C}^n$ contains a Zariski dense set of points with coordinates in $\overline{\mathbb{Q}}$, then Y is already defined over $\overline{\mathbb{Q}}$.

Proof of the Claim. Since $V(f_1, \ldots, f_r) = V(f_1) \cap \cdots \cap V(f_r)$, it is enough to consider the case $Y = V(f)$ for some $f \in \mathbb{C}[X_1, \ldots, X_n]$. Let $(b_i)_{i \in I}$ be a basis of the infinite dimensional $\overline{\mathbb{Q}}$-vector space $\mathbb{C}$. Writing every coefficient of f as a finite sum of basis elements, we obtain

$$f = \sum_{j=1}^{s} b_j g_j$$

with pairwise distinct b_j among the $(b_i)_{i \in I}$ and $g_j \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ for $j \in \{1, \ldots, s\}$. We claim that $V(f) = V(g_1, \ldots, g_s)$. Indeed, the inclusion $V(f) \supseteq V(g_1, \ldots, g_s)$ is immediate. For the other inclusion, first let $a \in V(f) \cap \overline{\mathbb{Q}}^n$. Then

$$\sum_{j=1}^{s} b_j g_j(a) = 0,$$

and we have $g_j(a) \in \overline{\mathbb{Q}}$. But the b_j are distinct basis elements, so $g_j(a) = 0$ for all $j \in \{1, \ldots, s\}$. Therefore, the $g_1, \ldots, g_s$ vanish on the dense subset $V(f) \cap \overline{\mathbb{Q}}^n$ of $V(f)$, and thus on all of $V(f)$. We conclude that $V(f) \subseteq V(g_1, \ldots, g_r)$ and hence $V(f) = V(g_1, \ldots, g_r)$. □(Claim)

The proof of the claim also gives that Y can be defined by finitely many polynomials in $\overline{\mathbb{Q}}[X_1, \ldots, X_n]$, which of course also directly follows from the fact that this ring is Noetherian. So Y is defined over the number field k obtained by adjoining the finitely many coefficients of these polynomials to $\mathbb{Q}$. □

Proposition 1.9. *For sufficiently large $t \in \mathbb{N}_{\geq 1}$ (in fact for all $t > 6$), if there is a point of height t in $X^{\text{trans}}(\mathbb{Q})$, then there are at least $\sqrt{t}/[k:\mathbb{Q}]$ points of height t in $X^{\text{trans}}(\mathbb{Q})$.*

Proof. Every field automorphism of $\overline{\mathbb{Q}} = \overline{k}$ fixing k permutes the zeros of polynomials with coefficients in k. Thus, since Y is defined over k, we get an action of $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ on Y by $\sigma.y := (\sigma(y_1), \ldots, \sigma(y_n))$. Furthermore, since all polynomials $X^\ell - 1$ for $\ell \in \mathbb{N}_{\geq 1}$ have coefficients in $\mathbb{Q} \subseteq k$, the action of $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ in each coordinate also preserves ℓ-th roots of unity. In fact, it even preserves primitive ℓ-th roots of unity. Indeed, let ζ be a primitive ℓ-th root of unity. Every automorphism $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)$ fixes $\mathbb{Q} \subseteq k$, hence permutes the zeros of the minimal polynomial $\mu_{\zeta,\mathbb{Q}}$ of ζ over $\mathbb{Q}$. But this polynomial splits over $\overline{\mathbb{Q}}$ as the product

$$\prod_{1 \leq m \leq \ell,\ \gcd(\ell,m)=1} (X - \zeta^m),$$

so its zeros in $\overline{\mathbb{Q}}$ are precisely all the primitive ℓ-th roots of unity. As claimed, σ must send primitive ℓ-th roots of unity to primitive ℓ-th roots of unity. Hence, we get an order-preserving action of $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ on $Y \cap \mathbb{G}^n_{\text{tor}}$. Given $(e^{2\pi i a_1/b_1}, \ldots, e^{2\pi i a_n/b_n}) \in Y \cap \mathbb{G}^n_{\text{tor}}$ with $0 \leq a_j < b_j$, a_j and b_j coprime and $a_j = 0$ for $b_j = 1$, as well as $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)$, we can write explicitly

$$\sigma.(e^{2\pi i a_1/b_1}, \ldots, e^{2\pi i a_n/b_n}) = (e^{2\pi i a'_1/b_1}, \ldots, e^{2\pi i a'_n/b_n})$$

for unique $a'_j \in \mathbb{N}$ such that $0 \leq a'_j < b_j$ with a'_j, b_j coprime and $a'_j = 0$ for $b_j = 1$. Through the bijection $X(\mathbb{Q}) \to Y \cap \mathbb{G}^n_{\text{tor}}$ the group $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ then also acts on $X(\mathbb{Q})$ by

$$\sigma.\left(\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\right) := \left(\frac{a'_1}{b_1}, \ldots, \frac{a'_n}{b_n}\right).$$

This action preserves the denominators, and hence also the heights. In the next subsection we will prove independently from the present proposition that $X^{\text{trans}} = X$ holds (1.15), so we obtain an action of $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ on $X^{\text{trans}}(\mathbb{Q})$.

Now assume that there is $x \in X^{\text{trans}}(\mathbb{Q})$ of height t. We write $x = (a_1/b_1, \cdots, a_n/b_n)$ with $a_j \in \mathbb{N}$ $b_j \in \mathbb{Z} \setminus \{0\}$ coprime and $b_j = 1$ if $a_j = 0$ for all $j \in \{1, \ldots, n\}$. As in our extended key observation (1.6) we then have $t = \max\{b_1, \ldots, b_n\}$ and each $e^{2\pi i a_j/b_j}$ is a primitive b_j-th root of unity. The height-preserving action described above yields that also the so-called Galois conjugates $\sigma.x$ of x for $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)$ are points in $X^{\text{trans}}(\mathbb{Q})$ of height t. To prove the proposition, it thus suffices to show that the set $\{\sigma.x : \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)\}$ has cardinality least $\sqrt{t}/[k:\mathbb{Q}]$. Take $j' \in \{1, \ldots, n\}$ such that $b_{j'}$ is maximal, i.e. such that $b_{j'} = t$. Then $e^{2\pi a_{j'}/b_{j'}}$ is a primitive t-th root of unity which we denote by ζ.

It is a basic result from Galois theory that given an algebraic field extension L/K and an intermediate field $K \subseteq Z \subseteq L$, then for every homomorphism $\sigma \colon Z \to \overline{K}$ fixing K there is a homomorphism $L \to \overline{K}$ extending σ. Applying this to $K = k$, $Z = k(\zeta)$ and $L = \overline{K} = \overline{\mathbb{Q}}$, it suffices to show that the set $\{\sigma.x : \sigma \in \mathrm{Aut}(k(\zeta)/k)\}$ has cardinality at least $\sqrt{t}/[k:\mathbb{Q}]$.

Every $\sigma \in \mathrm{Aut}(k(\zeta)/k)$ is uniquely determined by where it sends ζ, thus if $\sigma, \tau \in \mathrm{Aut}(k(\zeta)/k)$ are distinct then $\sigma(\zeta) \neq \tau(\zeta)$, and thus $\sigma.x \neq \tau.x$, as they differ in the j'-th coordinate. We therefore reduced our problem to showing that $\#\,\mathrm{Aut}(k(\zeta)/k) \geq \sqrt{t}/[k:\mathbb{Q}]$.

The field extension $k(\zeta)/k$ is separable as these fields are of characteristic zero. Moreover, $\mu_{\zeta,k}$ must divide $\mu_{\zeta,\mathbb{Q}}$, so all zeros of $\mu_{\zeta,k}$ in $\overline{\mathbb{Q}}$ are primitive roots of unity, which are already contained in $k(\zeta)$, so $k(\zeta)/k$ is also normal. Therefore, $k(\zeta)/k$ is Galois. We compute

$$\#\,\mathrm{Gal}(k(\zeta)/k) = [k(\zeta):k] = \frac{[k(\zeta):\mathbb{Q}]}{[k:\mathbb{Q}]} \geq \frac{[\mathbb{Q}(\zeta):\mathbb{Q}]}{[k:\mathbb{Q}]}.$$

But $[\mathbb{Q}(\zeta):\mathbb{Q}]$ is precisely the number of distinct t-th roots of unity, that is $\varphi(t)$, where $\varphi\colon \mathbb{N}_{\geq 1} \to \mathbb{N},\ t \mapsto \#(\mathbb{Z}/t\mathbb{Z})^\times$ is the Euler totient function. We have the elementary and very rough estimate $\varphi(t) \geq \sqrt{t}$ if $t > 6$, and conclude that

$$\#\,\mathrm{Gal}(k(\zeta)/k) \geq \frac{[\mathbb{Q}(\zeta):\mathbb{Q}]}{[k:\mathbb{Q}]} \geq \frac{\sqrt{t}}{[k:\mathbb{Q}]}$$

if $t > 6$, completing the proof of the proposition. □

We can now compare the bounds from Pila-Wilkie (1.7) and the preceding proposition (1.9).

Corollary 1.10. *The set $X^{\mathrm{trans}}(\mathbb{Q})$ is finite. In particular, there are only finitely many points in its image $E(X^{\mathrm{trans}}(\mathbb{Q})) \subseteq \mathbb{G}^n_{\mathrm{tor}}$.*

Proof. It suffices to show that the height of points in $X^{\mathrm{trans}}(\mathbb{Q})$ is bounded. To be explicit, take $\varepsilon = 1/3$ in Pila-Wilkie (1.7) and let C be the corresponding constant. So for every $t \in \mathbb{R}_{\geq 0}$ there are at most $Ct^{1/3}$ points of height up to t in $X(\mathbb{Q})$.

Let $x \in X^{\mathrm{trans}}(\mathbb{Q})$ be a point of height $\mathsf{H}(x) = t \in \mathbb{N}_{>6}$. By the previous proposition (1.9), there are at least $t^{1/2}/[k:\mathbb{Q}]$ points of exact height t in $X^{\mathrm{trans}}(\mathbb{Q})$. By Pila-Wilkie, there exist at most $Ct^{1/3}$ points of height up to t in $X^{\mathrm{trans}}(\mathbb{Q})$. We obtain the inequality

$$\frac{t^{1/2}}{[k:\mathbb{Q}]} \leq Ct^{1/3},$$

which holds if and only if $t \leq (C[k:\mathbb{Q}])^6$. As a result, the height of rational points in X^{trans} is bounded by $\max\{6, (C[k:\mathbb{Q}])^6\}$. □

1.3 Functional transcendence

This final subsection on Mann's theorem concerns the algebraic part of X. The central ingredient will be a power series version of Schanuel's conjecture, which was proved by Ax in 1971 [Ax71].

Definition 1.11. Let V be a K-vector space and let $W \subseteq V$ be a K-subspace. Then $v_1, \ldots, v_n \in V$ are **linearly independent over K modulo W** if for all $a_1, \ldots, a_n \in K$, we have that $\sum_{j=1}^n a_j v_j \in W$ implies $a_1 = \cdots = a_n = 0$.

Theorem 1.12 (Cor. 1 in [Ax71])**.** *Let $\gamma_1, \ldots, \gamma_n \in \mathbb{C}[\![t]\!]$ be complex power series with no constant term that are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$. Then the transcendence degree of*

$$\mathbb{C}(\gamma_1, \ldots, \gamma_n, \exp(\gamma_1), \ldots, \exp(\gamma_n))$$

over $\mathbb{C}$ is at least $n+1$.

We will elaborate a little more on the background of this theorem later on (Subsection 4.2).

Definition 1.13. Let $n \in \mathbb{N}_{\geq 1}$ and let $Y \subseteq \mathbb{C}^n$ be an algebraic set. The **special locus** of Y in $\mathbb{G}^n$, denoted by $\mathrm{SpL}^n(Y)$, is defined as the union

$$\bigcup_{(a,H)\in\mathcal{S}} aH \subseteq Y,$$

where $\mathcal{S}$ is the set of all pairs (a, H) with $a \in \mathbb{G}^n$ and H an infinite algebraic subgroup of $\mathbb{G}^n$ such that $aH \subseteq Y$.

Remark. With the present assumption of Mann's Theorem, we have $\mathrm{SpL}^n(Y) = \emptyset$.

We will now argue that X^{alg} corresponds to this special locus under E. Unfortunately, the proof of this requires some machinery that is not elementary enough to be discussed on the spot. Therefore, the proof will be less detailed, hopefully conveying the central ideas nevertheless. We refer to Humphrey's book for relevant results on linear algebraic groups [Hum75].

Proposition 1.14. *We have $E(X^{\mathrm{alg}}) \subseteq \mathrm{SpL}^n(Y)$.*

Proof. For $Y = \mathbb{C}^n$ we get $\mathrm{SpL}^n(Y) = \mathbb{G}^n$, and for $Y = \emptyset$ we get $E(X^{\mathrm{alg}}) = \emptyset$. Thus, we may assume that $\emptyset \subsetneq Y \subsetneq \mathbb{C}^n$. As remarked after the definition of the algebraic part (1.5), at the cost of missing out finitely many points, we may suppose that

$$X^{\mathrm{alg}} = \bigcup_{\substack{\gamma\colon(-1,1)\to X \text{ real analytic,} \\ \text{semialgebraic and nonconstant}}} \mathrm{im}(\gamma)$$

so it suffices to prove the following:

Claim. Let $Y \subseteq \mathbb{C}^n$ be a proper nonempty algebraic set. Every real analytic semialgebraic nonconstant map $\gamma\colon (-1,1) \to X$ satisfies $E(\mathrm{im}(\gamma)) \subseteq \mathrm{SpL}^n(Y)$.

Proof of the Claim. Let $\gamma\colon (-1,1) \to X \subseteq \mathbb{R}^n$ be a real analytic semialgebraic nonconstant map. By analytic continuation, we find $\delta > 0$ such that γ extends to a complex analytic map on $(-1,1) + i(-\delta,\delta) \to \mathbb{C}^n$. We proceed by induction on n.

For the base case, let $n = 1$. Then the algebraic set $Y \subsetneq \mathbb{C}$ is a finite number of points, and we also know that $E\colon \mathbb{C} \to \mathbb{G},\ z \mapsto e^{2\pi i z}$ is a local homeomorphism, so $E^{-1}(Y)$ is discrete. Thus, any continuous function $\gamma\colon (-1,1) \to X \subseteq E^{-1}(Y)$ is constant and the claim holds.

Let $n > 1$ and suppose that the claim holds for $n-1$. Choosing a different basis if necessary, we write γ in coordinates $(\gamma_1, \ldots, \gamma_n)$ where $\gamma_j(0) = 0$ for all $j \in \{1, \ldots, n\}$, so that the respective power series have no constant term. We have $\mathrm{im}(\gamma) \subseteq X \subseteq E^{-1}(Y)$, and thus

$$(e^{2\pi i\gamma_1(t)}, \ldots, e^{2\pi i\gamma_n(t)}) \in Y$$

for all $t \in (-1,1)$. Let $f_1, \ldots, f_r$ be the polynomials whose zero set is Y. Since $Y \subsetneq \mathbb{C}^n$, there is $\ell \in \{1, \ldots, r\}$ such that f_ℓ is nonzero. We then have

$$f_\ell(e^{2\pi i\gamma_1(t)}, \ldots, e^{2\pi i\gamma_n(t)}) = 0$$

for all $t \in (-1,1)$, hence also for all $t \in (-1,1) + i(-\delta,\delta)$ by the identity theorem. We conclude that the functions $e^{2\pi i\gamma_1}, \ldots, e^{2\pi i\gamma_n}$ are algebraically dependent over $\mathbb{C}$, hence so are the respective complex power series. Moreover, as γ is semialgebraic, the transcendence degree of $\mathbb{C}(2\pi i\gamma_1, \ldots, 2\pi i\gamma_n)$ over $\mathbb{C}$ can be seen to be 1. Therefore, the transcendence degree of

$$\mathbb{C}(2\pi i\gamma_1, \ldots, 2\pi i\gamma_n, e^{2\pi i\gamma_1}, \ldots, e^{2\pi i\gamma_n})$$

over $\mathbb{C}$ is smaller or equal then $1 + (n-1) = n$. It follows from Ax's theorem (1.12) that the $2\pi i\gamma_1, \ldots, 2\pi i\gamma_n$ are linearly dependent over $\mathbb{Q}$ modulo $\mathbb{C}$, say $\sum_{j=1}^n a_j 2\pi i\gamma_j = w \in \mathbb{C}$, with $a_j \in \mathbb{Q}$ for $j \in \{1, \ldots, n\}$, some of which are nonzero. So the image of γ is contained in the zero set of the linear polynomial $\sum_{j=1}^n 2\pi i a_j X_j - w$. This is a translate of an $(n-1)$-dimensional linear subspace of $\mathbb{C}^n$, so its image under E will be a translate of an $(n-1)$-dimensional algebraic subgroup of $\mathbb{G}^n$, as we can prove with more detail:

Subclaim. Let $h = \sum_{j=1}^n 2\pi i a_j X_j - w$ be a linear polynomial with $a_j \in \mathbb{Q}$ for $j \in \{1, \ldots, n\}$, some of which are nonzero, and $w \in \mathbb{C}$. Then the vanishing set $V(h) \subseteq \mathbb{C}^n$ is a translate of an $(n-1)$-dimensional linear subspace U of $\mathbb{C}^n$ and $E(V(h))$ is the translate of an $(n-1)$-dimensional algebraic subgroup of $\mathbb{G}^n$.

Proof of the Subclaim. We can multiply h with the least common multiple of the denominators of the a_j to obtain $\tilde{h} = \sum_{j=1}^{n} 2\pi i m_j X_j - \tilde{w}$ with $m_j \in \mathbb{Z}$ for $j \in \{1, \ldots, n\}$ and $\tilde{w} \in \mathbb{C}$. This does not change the zero set, i.e. $V(h) = V(\tilde{h})$. We define

$$M_+ \coloneqq \{j \in \{1, \ldots, n\} : m_j > 0\} \quad \text{and} \quad M_- \coloneqq \{j \in \{1, \ldots, n\} : m_j < 0\}.$$

Replacing $\tilde{w}$ with $\tilde{w}/(2\pi i \#(M_+ \cup M_-))$, we can write $\tilde{h} = \sum_{j \in M_+ \cup M_-} 2\pi i m_j (X_j - \tilde{w}/m_j)$, so $V(h)$ is the translate of the vector subspace

$$U \coloneqq V(\sum_{k=1}^{n} 2\pi i m_k X_k)$$

of $\mathbb{C}^n$ by the vector $\widehat{w} = (\widehat{w}_1, \ldots, \widehat{w}_n)$ where $\widehat{w}_k$ for $k \in \{1, \ldots, n\}$ is $\tilde{w}/m_k$ if $m_k \neq 0$ and 0 otherwise. We obtain

$$\begin{aligned}
E(V(h)) &= E(V(\tilde{h})) \\
&= \{(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}) : (z_1, \ldots, z_n) \in \mathbb{C}^n,\ \textstyle\sum_{j \in M_+ \cup M_-} 2\pi i m_j (z_j - \frac{\tilde{w}}{m_j}) = 0\} \\
&= \{(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}) : (z_1, \ldots, z_n) \in \mathbb{C}^n,\ \exp(\textstyle\sum_{j \in M_+ \cup M_-} 2\pi i m_j (z_j - \frac{\tilde{w}}{m_j})) = 1\} \\
&= \{(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}) : (z_1, \ldots, z_n) \in \mathbb{C}^n,\ \textstyle\prod_{j \in M_+ \cup M_-} (e^{2\pi i z_j} e^{-2\pi i \tilde{w}/m_j})^{m_j} = 1\} \\
&= \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{G}^n : \textstyle\prod_{j \in M_+} (e^{-2\pi i \tilde{w}/m_j} \zeta_j)^{m_j} - \prod_{j \in M_-} (e^{-2\pi i \tilde{w}/m_j} \zeta_j)^{-m_j} = 0\} \\
&= (a_1, \ldots, a_n) \cdot \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{G}^n : \textstyle\prod_{j \in M_+} \zeta_j^{m_j} - \prod_{j \in M_-} \zeta_j^{-m_j} = 0\}
\end{aligned}$$

where a_j for $j \in \{1, \ldots, n\}$ is $e^{2\pi i \tilde{w}/m_j}$ if $m_j \neq 0$ and 1 otherwise. As a result, we can write $E(V(\tilde{h}))$ as the translate of the Zariski closed subset

$$H \coloneqq \mathbb{G}^n \cap V(\textstyle\prod_{j \in M_+} X_j^{m_j} - \prod_{j \in M_-} X_j^{-m_j})$$

of $\mathbb{G}^n$ by $a \coloneqq (a_1, \ldots, a_n) \in \mathbb{G}^n$. Moreover, H is in fact a subgroup of $\mathbb{G}^n$. This can be seen directly from the description or from the fact that H corresponds to the image of a zero set of a linear polynomial with zero constant term under E, i.e. to the image of a vector subspace of $\mathbb{C}^n$ under E, and that E maps vector subspaces of $\mathbb{C}^n$ to subgroups of $\mathbb{G}^n$. Furthermore, the algebraic set $V(\prod_{j \in M_+} X_j^{m_j} - \prod_{j \in M_-} X_j^{-m_j})$ has codimension 1 in $\mathbb{C}^n$. □(Subclaim)

We now know that $E(V(h)) = aH$ for some $(n-1)$-dimensional algebraic subgroup H of $\mathbb{G}^n$ and $a \in \mathbb{G}^n$. We use a nontrivial fact from the theory of linear algebraic groups: Every connected algebraic subgroup of $\mathbb{G}^n$ is isomorphic to $\mathbb{G}^{n'}$ for some $n' \leq n$ (cf. Theorem in 16.2 in [Hum75]). While H might not be connected, its identity component, i.e. the connected component of H that contains the neutral element, is connected. Furthermore, every other connected component is a translate of this identity component (cf. 7.3. in [Hum75]). Hence, by replacing H with its identity component, we may assume that $\mathbb{G}^{n-1} \cong H$. With notation as in the proof of the Subclaim, our situation can be displayed in the following diagram

$$\begin{tikzcd}
 & & (-1,1) \arrow[d] \arrow[dr, "\gamma"] & \\
\mathbb{C}^{n-1} \arrow[r, "\cong"] \arrow[d, "E'"] & U \arrow[r, "\widehat{w}+"] & \widehat{w} + U \arrow[r, hook] \arrow[d] & \mathbb{C}^n \arrow[d, "E"] \\
\mathbb{G}^{n-1} \arrow[r, "\cong"] & H \arrow[r, "a\cdot"] & aH \arrow[r, hook] & \mathbb{G}^n.
\end{tikzcd}$$

By scaling with a^{-1} and then applying the inverse of the isomorphism $\mathbb{G}^{n-1} \to H$, we can view $Y \cap aH$ as a Zariski closed subset of $\mathbb{G}^{n-1}$ and γ as a map to the preimage of $Y \cap aH$. Identifying $\mathbb{G}^{n-1}$ with the algebraic subgroup H of $\mathbb{G}^n$, the special locus of $Y \cap aH$

in $\mathbb{G}^{n-1}$ is then contained the special locus of Y in $\mathbb{G}^n$ by definition. As a result, we obtain $E(\mathrm{im}(\gamma)) \subseteq \mathrm{SpL}^{n-1}(Y \cap aH) \subseteq \mathrm{SpL}^n(Y)$ by induction as desired. □(Claim)

This completes the proof of Proposition 1.14. □

Under the assumption of Mann's Theorem, we have $\mathrm{SpL}(Y) = \emptyset$, so by the previous proposition (1.14) also $X^{\mathrm{alg}} = \emptyset$. This means that $X = X^{\mathrm{trans}}(\mathbb{Q})$. As promised in the previous subsection, we obtain the following:

Corollary 1.15. *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ *on* $X(\mathbb{Q})$ *in the proof of Proposition 1.9 is in fact an action on* $X^{\mathrm{trans}}(\mathbb{Q})$.

Let us conclude the proof of Mann's Theorem.

Proof of Mann's Theorem (1.1). We assume that the algebraic set $Y \subseteq \mathbb{C}^n$ does not contain any translate of an infinite closed algebraic subgroup of $\mathbb{G}^n$ and want to show that $Y \cap \mathbb{G}^n_{\mathrm{tor}}$ is finite. We have

$$Y \cap \mathbb{G}^n_{\mathrm{tor}} = E(X^{\mathrm{trans}}(\mathbb{Q})) \cup E(X^{\mathrm{alg}}(\mathbb{Q})).$$

We proved with bounds from o-minimality and Galois theory that $E(X^{\mathrm{trans}}(\mathbb{Q}))$ is finite (1.10). Moreover, $E(X^{\mathrm{alg}}(\mathbb{Q})) = E(X^{\mathrm{alg}}) \cap \mathbb{G}^n_{\mathrm{tor}}$ is contained in $\mathrm{SpL}(Y)$ by the previous proposition (1.14). However, by the present assumption of Mann's Theorem, $\mathrm{SpL}(Y)$ is empty, hence also $E(X^{\mathrm{alg}}(\mathbb{Q}))$ is empty. We conclude that $Y \cap \mathbb{G}^n_{\mathrm{tor}}$ is finite. □

2 Elementary theory of Abelian varieties

In this section, we introduce Abelian varieties, which are the central objects to the statement of Manin-Mumford. After a very brief subsection on *abstract algebraic varieties*, we discuss *complete* varieties. Completeness is a main feature of Abelian varieties that distinguishes them from other algebraic groups. In a third subsection, we finally introduce *Abelian varieties* and prove basic properties. A final part is devoted to *complex Abelian varieties* and the connections between the Zariski topology and the Euclidean topology on them.

Throughout, let k denote an algebraically closed field.

2.1 Review of algebraic varieties

We should clarify what we mean by an *algebraic variety*, for which there are many approaches.

From the viewpoint of basic model theory, algebraic sets are what we want to work with. They are precisely the sets definable by quantifier- and negation-free formulas in $(k, +, -, \cdot, 0, 1)$. Also projective algebraic sets can often be studied conveniently with basic model theory by considering their affine cones. From the viewpoint of algebraic geometry, however, the fact that algebraic sets are always embedded into some k^n is considered a disadvantage, an obstruction from studying them intrinsically. If we were to look at varieties without sheaves, we would narrow our view too much, while defining them as reduced schemes of finite type over k would make it a bit cumbersome to translate back to model theory.

We will try to strike a balance by considering *abstract algebraic varieties in the sense of FAC*, i.e. varieties as Serre defined them in *Faisceaux algébriques cohérents* [Ser55]. These are not defined as schemes, but nevertheless as spaces equipped with a sheaf. An instructive treatment of them can be found in lecture notes by Milne, on which much of this section is based [Mil17]. Let us briefly recall some definitions, of course not remotely presenting all of the theory.

Definition 2.1. For an open subset U of an algebraic set V, we call a function $f: U \to k$ **regular** if for all points $p \in U$ there exist g and h in the coordinate ring $k[V]$ of V with $h(p) \neq 0$ such that $f = g/h$ in some open neighbourhood of p.

The set of regular functions on an open subset U of V is naturally a k-algebra, denoted by $\mathcal{O}_V(U)$, and the functor $\mathsf{Ouv}(V) \to k\text{-}\mathsf{Alg}$, $U \mapsto \mathcal{O}_V(U)$ is a sheaf of k-algebras on V, where $\mathsf{Ouv}(V)$ denotes the category of open subsets of V with reversed inclusions as morphisms. This gives $(V, \mathcal{O}_V)$ the structure of a k-ringed space (cf. Prop. 3.9. in [Mil17]).

Definition 2.2. An **affine variety** over k is a k-ringed space isomorphic to $(V, \mathcal{O}_V)$, where V is an algebraic set and $\mathcal{O}_V$ assigns to an open subset $U \subseteq X$ the regular functions on U.

A central subject of *analytic geometry* is not the study of open subsets in some $\mathbb{C}^n$ but of objects that are just locally open subsets of $\mathbb{C}^n$. We could define a complex analytic manifold as a $\mathbb{C}$-ringed space $(M, \mathcal{O}_M)$ such that M is Hausdorff and every point in M has an open neighbourhood U such that $(U, \mathcal{O}_M|U)$, where $\mathcal{O}_M|U$ denotes the restriction, is isomorphic as a $\mathbb{C}$-ringed space to the $\mathbb{C}$-ringed space of an open subset of $\mathbb{C}^n$ with its sheaf of holomorphic functions. This motivates the definition of *algebraic (pre)varieties*, which are objects that are *locally* affine but may *globally* look different.

Definition 2.3. An **(algebraic) prevariety** over k is a k-ringed space $(V, \mathcal{O}_V)$ such that V is quasi-compact and for every point P in V there is an open neighbourhood U of P such that $(U, \mathcal{O}_V|U)$ is an affine variety.

Remark. By quasi-compactness, every algebraic prevariery $(V, \mathcal{O}_V)$ admits a finite open cover $V = \bigcup_{i=1}^n U_i$ by affine varieties $(U_i, \mathcal{O}_V|U_i)$. We call such opens the *open affines*.

The definition of algebraic prevarieties still allows some pathological objects such as the line with two origins.

In the Euclidean setting, we can impose the Hausdorff condition to avoid such examples. The Zariski topology is way too coarse to be Hausdorff, so we need a substitute. Recall that a continuous map into a Hausdorff space is determined by its values on a dense subset. This motivates the following definition.

Definition 2.4. An algebraic prevariety V is **separated** if for every algebraic prevariety W and any two morphisms of k-ringed spaces $\mu_1, \mu_2 \colon W \to V$, the set $\{w \in W : \mu_1(w) = \mu_2(w)\}$ is closed in W. An **(algebraic) variety** over k is a separated prevariety over k.

A pleasant feature of these definitions is that the respective morphisms can be defined concisely.

Definition 2.5. A map of affine varieties, prevarieties or varieties over k is **regular** if it is a morphisms of k-ringed spaces (cf. section 3.d. in [Mil17]).

In terms of category theory, we have full subcategories

$$\mathsf{affine\ varieties} \subsetneq \mathsf{algebraic\ varieties} \subsetneq \mathsf{prevarieties} \subsetneq k\text{-}\mathsf{ringed\ spaces}.$$

As Abelian varieties are projective, the most important examples of algebraic varieties for us will be the projective space and its closed subvarieties.

Construction 2.6. The underlying set of the **projective n-space** over k, denoted by $\mathbb{P}^n(k)$, is the set of orbits

$$\mathbb{P}^n(k) := (k^{n+1} \setminus \{0\})/k^\times$$

of $k^{n+1} \setminus \{0\}$ under the action of $k^\times$ by coordinatewise multiplication. We topologise $\mathbb{P}^n(k)$ by defining its closed subsets to be the zero sets of homogeneous polynomials in $k[X_0, \ldots, X_n]$. For $i \in \{0, \ldots, n\}$, the open subsets

$$U_i := \{[x_0 : \ldots : x_n] \in \mathbb{P}^n(k) : x_i \neq 0\}$$

of $\mathbb{P}^n(k)$ form a finite open cover $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$. We have maps to the affine n-space $\mathbb{A}^n(k) = k^n$ with its sheaf of regular functions given by

$$\mu_i \colon U_i \to \mathbb{A}^n(k), \quad [x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

and it can be proved that there is a unique structure of an algebraic variety on $\mathbb{P}^n(k)$ such that the U_i are the open affines with each μ_i yielding an isomorphism of varieties, cf. Proposition 6.12 in [Mil17]. By a **projective variety**, we will always mean a closed subvariety of $\mathbb{P}^n(k)$, cf. section 5.e. in [Mil17].

There are many more topics that will be used in our following discussions, but for which an adequate treatment would be too lengthy. Even constructions such as *subvarieties* or *products of varieties* require some effort to be carried out properly, cf. sections 5.e,f,g in [Mil17]. Purely topological concepts such as *irreducibility* and *Krull dimension* can be defined for abstract varieties as in the case of algebraic sets. For *smoothness* and *tangent spaces*, we refer to section 4 in [Mil17]. For *extensions of scalars* and varieties over non algebraically closed fields, we refer to section 11 in [Mil12]. The special case of extension of scalars from $\mathbb{R}$ to $\mathbb{C}$ is presented in much detail in the book by Mangolte [Man20].

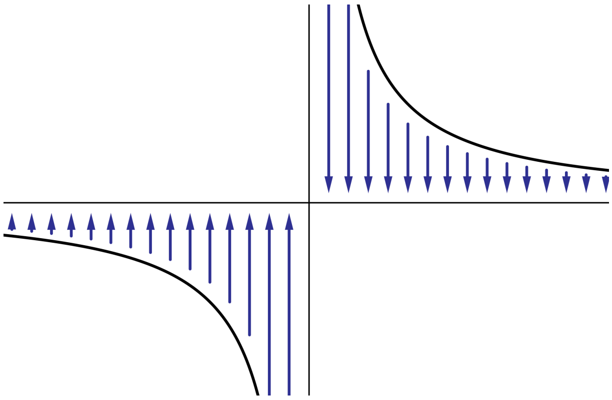
2.2 Complete varieties

In topology, we learn that compact manifolds are exceptionally well-behaved. Completeness is for algebraic geometers what compactness is for analytic geometers. Recall that a space X is quasi-compact if and only if for all spaces Y, the projection $\mathrm{pr}_2\colon X \times Y \to Y$ is a closed map.

Definition 2.7. An algebraic variety X is **complete** if for all algebraic varieties Y, the projection $\mathrm{pr}_2\colon X \times Y \to Y$ is a closed map.

Because the Zariski topology on $X \times Y$ usually does not agree with the product topology, this is not equivalent to quasi-compactness of Y (which holds by the definition of varieties).

Example 2.8. The affine line $\mathbb{A}^1(\mathbb{C})$ is not complete. Consider the algebraic set $V(XY-1)$ in $\mathbb{A}^2(\mathbb{C}) \cong \mathbb{A}^1(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$. Under the projection $\mathrm{pr}_2\colon \mathbb{A}^1(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$ it is mapped to $\mathbb{A}^1(\mathbb{C}) \setminus \{0\}$, which is not closed.



Let us summarise some basic results:

Lemma 2.9 (7.3.-7.11. in [Mil17])**.**

(a) *Closed subvarieties of complete varieties are complete.*

(b) *A variety is complete if and only if its irreducible components are complete.*

(c) *Finite products of complete varieties are complete.*

(d) *If $\mu\colon X \to Y$ is a regular map of varieties and X is complete, then the image of μ is a complete closed subvariety of Y.*

(e) *A regular map from a complete connected variety to an affine variety is constant. In particular, every complete connected affine variety is a point.*

(f) *A variety X is complete if and only if for every affine variety Y the map $\mathrm{pr}_2\colon X \times Y \to Y$ is closed. In fact, X is complete if and only if $\mathrm{pr}_2\colon X \times \mathbb{A}^n(k) \to \mathbb{A}^n(k)$ is closed.*

We will now show that every projective variety is complete. There are various proofs known for this. We will give a model theoretic one due to van den Dries [vdD82]. For a purely algebraic proof, see Theorem 7.22. in [Mil17].

Theorem 2.10. *Every projective variety is complete.*

The proof can be found below, we first present the required background from model theory and valuation theory. We will reduce the statement to showing that a certain subset of $\mathbb{A}^n(k)$ is closed, i.e. an affine algebraic set. The affine algebraic sets are precisely the sets definable by positive quantifier free formulas in $(k,+,-,\cdot,0,1)$. Therefore, we will need a *positive*, i.e. negation-free, version of quantifier elimination. The following result, called the *Lyndon-Robinson type Lemma*, can be proved from the compactness theorem.

Lemma 2.11 (Exercise 3.4.21 in [Mar02] and the first lemma in [vdD82])**.** *Let $\mathcal{L}$ be a language that contains a constant symbol. Let T be an $\mathcal{L}$-theory and let $\varphi(\vec{x})$ be an $\mathcal{L}$-formula. Suppose that for all models $\mathcal{K}$ and $\mathcal{K}'$ of T, every $\mathcal{L}$-substructure $\mathcal{A}$ of $\mathcal{K}$ and each $\mathcal{L}$-homomorphism $\sigma\colon \mathcal{A} \to \mathcal{K}'$ we have the following:*

$$\text{If } \vec{a} \in A^n \text{ and } \mathcal{K} \models \varphi(\vec{a}), \text{ then } \mathcal{K}' \models \varphi(\sigma(\vec{a})).$$

Then there exists a positive quantifier free formula $\psi(\vec{x})$ such that $T \models \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$.

Remark. We only need the assumption that the language contains a constant symbol in the case where φ is an $\mathcal{L}$-sentence.

Recall that a *valuation ring* of a field k is a subring $\mathcal{O}$ of k such that for all $a \in k$ we have $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$. The following results can be found in Lang's *Algebra* [Lan02].

Lemma 2.12 (VII §3 Corollary 3.3. in [Lan02])**.** *Let k be a field and let $\sigma\colon A \to k'$ be a ring homomorphism of a subring A of k into an algebraically closed field k'. Then σ can be extended to a ring homomorphism $\mathcal{O} \to k'$, where $\mathcal{O}$ is a valuation ring of k.*

Lang's formulation of the next lemma is slightly different, but his discussion proceeding it shows that our formulation is equivalent, cf. page 349 in [Lan02].

Lemma 2.13 (VII §3 Proposition 3.4. in [Lan02])**.** *Let $\mathcal{O}$ be a valuation ring of k. Let $q \in \mathbb{N}$. Given finitely many nonzero elements $b_0, \dots, b_q \in k$, there exists an index $j \in \{0, \dots, q\}$ such that $b_p/b_j \in V$ for all $p \in \{0, \dots, q\}$.*

With these three lemmas at hand, we can prove that projective varieties are complete.

Proof of Theorem 2.10. We have to show that for a projective variety X and any algebraic varitey Y, the projection $\mathrm{pr}_2\colon X \times Y \to Y$ is closed. Because closed subvarieties of complete varieties are complete (2.9(a)), it is enough to consider $X = \mathbb{P}^n(k)$. We can also assume $Y = \mathbb{A}^m(k)$ (2.9(f)), so it remains to prove that $\mathrm{pr}_2\colon \mathbb{P}^n(k) \times \mathbb{A}^m(k) \to \mathbb{A}^m(k)$ is closed.

Let $Z \subseteq \mathbb{P}^n(k) \times \mathbb{A}^m(k)$ be closed. We show that $\mathrm{pr}_2(Z)$ is closed in $\mathbb{A}^n(k)$. We find polynomials

$$f_1, \dots, f_r \in k[X_0, \dots, X_n, Y_1, \dots, Y_m]$$

homogeneous in $X_0, \dots, X_n$ whose zero set in $\mathbb{P}^n(k) \times \mathbb{A}^n(k)$ is Z. We first assume that the f_i have integer coefficients. Let the homogeneous degree of f_i in $X_1, \dots, X_n$ be d_i and write

$$f_i = \sum_{\beta \in \mathbb{N}^m, |\beta| = d_i} g_{i,\beta} X^\beta$$

with $g_{i,\beta} \in \mathbb{Z}[Y_1, \dots, Y_m]$ for all $i \in \{1, \dots, r\}$ and all $\beta \in \mathbb{N}^m$. Now $\mathrm{pr}_2(Z)$ is the subset of k^m defined by the $\mathcal{L}_{\mathrm{ring}}$-formula

$$\exists x_0 \cdots \exists x_n \left(\bigvee_{j=0}^{n} x_j \neq 0 \wedge \bigwedge_{i=1}^{r} \sum_{\beta \in \mathbb{N}^m, |\beta| = d_i} g_{i,\beta}(y_1, \dots, y_m) x_0^{\beta_1} \cdots x_n^{\beta_n} = 0 \right).$$

Let us call this formula $\varphi(y_1, \dots, y_m)$. We have to show that φ is equivalent to a positive quantifier free $\mathcal{L}_{\mathrm{ring}}$-formula. We check the assumptions of Lyndon-Robinson (2.11), considering $\mathcal{L}_{\mathrm{ring}}$-theory of algebraically closed fields.

Let k and k' be algebraically closed fields, let $A \subseteq k$ be a subring and let $\sigma\colon A \to k'$ be a ring homomorphism. Suppose that there is $a \in k^m$ such that $\varphi(a)$ holds in k. Let $b \in k^{n+1}$ witness that $\varphi(a)$ holds in k, so $b \neq 0$ and for all $i \in \{1, \dots, r\}$ we have

$$\sum_{\beta \in \mathbb{N}^m, |\beta| = d_i} g_{i,\beta}(a_1, \dots, a_m) b_0^{\beta_1} \cdots b_n^{\beta_n} = 0.$$

We find a valuation ring $\mathcal{O}$ of k containing A and can extend σ to $\tilde{\sigma}\colon \mathcal{O} \to k'$ (2.12). Let $b_{\ell_0}, \dots, b_{\ell_q}$ be the finitely many nonzero coordinates of $b = (b_0, \dots, b_n)$. Since $b \neq 0$, we have $q \geq 0$ and thus find $j \in \{\ell_0, \dots, \ell_q\}$ such that $b_{l_p}/b_j \in \mathcal{O}$ for all $p \in \{0, \dots, q\}$ (2.13). Let $d = (d_0, \dots, d_n) \coloneqq (b_0/b_j, \dots, b_n/b_j)$ and $d' \coloneqq \tilde{\sigma}(d) \in k'$. We claim that $\varphi(\sigma(a))$ holds in k', witnessed by d'. Since the j-th entry of d is 1, so also is j-th entry of d' and one of the entries of d' is nonzero. The $g_{i,\beta}$ have integer coeffcients and σ is a ring homomorphism, thus

$$\sum_{\beta \in \mathbb{N}^m, |\beta| = d_i} g_{i,\beta}(\sigma(a_1), \dots, \sigma(a_m)) {d'}_0^{\beta_1} \cdots {d'}_n^{\beta_n} = 0$$

for all $i \in \{1, \dots, r\}$. Indeed, $\varphi(\sigma(a))$ holds in k', witnessed by d'.

For the general case, now let $f_1, \dots, f_r$ have any coefficients in k. Let $(\mu_i)_{i=1}^N$ be the these finitely many coefficients. Replacing each μ_i by a new variable W_i, we obtain polynomials

$$\tilde{f}_1, \dots, \tilde{f}_r \in \mathbb{Z}[X_0, \dots, X_n, Y_1, \dots, Y_m, W_1, \dots, W_N],$$

which are homogeneous in the $X_0, \dots, X_n$. Let $\tilde{Z}$ denote their vanishing set in $\mathbb{P}^n(k) \times \mathbb{A}^{m+N}(k)$. Consider the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^n(k) \times \mathbb{A}^m(k)) \times \mathbb{A}^N(k) \arrow[r, "\mathrm{pr}_2 \times \mathrm{id}"] \arrow[d, "p"'] & \mathbb{A}^m(k) \times \mathbb{A}^N(k) \arrow[d, "p'"] \\
\mathbb{P}^n(k) \times \mathbb{A}^m(k) \arrow[r, "\mathrm{pr}_2"'] & \mathbb{A}^m(k)
\end{tikzcd}$$

where the vertical maps p and p' are the projection maps to the first factor (as indicated by the parentheses). By the statement for integer coefficients, the map $\mathrm{pr}_2 \times \mathrm{id}$ is closed. The projection p restricts to a homeomorphism $V((W_i - \mu_i)_{i=1}^N) \to \mathbb{P}^n(k) \times \mathbb{A}^m(k)$ under which the closed set $\tilde{Z} \cap V((W_i - \mu_i)_{i=1}^N)$ is mapped to Z. Moreover, we have

$$(\mathrm{pr}_2 \times \mathrm{id})(\tilde{Z} \cap V((W_i - \mu_i)_{i=1}^N)) = (\mathrm{pr}_2 \times \mathrm{id})(\tilde{Z}) \cap V((W_i - \mu_i)_{i=1}^N),$$

and this set is closed in $\mathbb{A}^m(k) \times \mathbb{A}^N(k)$ because $(\mathrm{pr}_2 \times \mathrm{id})(\tilde{Z})$ is closed. Similarly, p' restricts to a homeomorphism $V((W_i - \mu_i)_{i=1}^N) \to \mathbb{A}^m(k)$ under which $(\mathrm{pr}_2 \times \mathrm{id})(\tilde{Z}) \cap V((W_i - \mu_i)_{i=1}^N)$ is mapped to $\mathrm{pr}_2(Z)$. We conclude that $\mathrm{pr}_2(Z)$ is closed. $\square$

Remark. The intermediate result we proved for polynomials with integer coefficients is called the *main theorem of elimination theory*: Let $f_1, \dots, f_r \in \mathbb{Z}[X_0, \dots, X_n, Y_1, \dots, Y_m]$ be homogeneous in the $X_0, \dots, X_n$. Then there exist polynomials $g_1, \dots, g_s \in \mathbb{Z}[Y_1, \dots, Y_n]$ such that for all $a \in k^m$ the system $f_1(X, a) = \dots = f_r(X, a) = 0$ has a nontrivial solution in k if and only if $g_1(a) = \dots = g_s(a) = 0$. You can ask for good algorithms to compute the g_i, and so elimination theory regained popularity with the rise of computers. Although the presented proof requires the axiom of choice, it suggests that questions of complexity in elimination theory can be related to the complexity of positive quantifier elimination.

We now know a large and important class of complete varieties. Coming back to their properties, the following theorem will be the key ingredient for understanding the group structure of Abelian varieties.

Theorem 2.14 (Rigidity Theorem)**.** *Let V, W and T be algebraic varieties such that V is complete and irreducible and such that W irreducible. Let $\varphi\colon V \times W \to T$ be a regular map. Suppose that there is $w_0 \in W$ such that $\varphi(-, w_0)\colon V \to T$ is constant. Then there is a regular map $g\colon W \to T$ such that $\varphi(v, w) = g(w)$ for all $v \in V$ and $w \in W$, i.e. such that the diagram*

$$\begin{tikzcd}
V \times W \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_2"'] & & T \\
& W \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

commutes.

Proof following 7.34. in [Mil17]. Since irreducible varieties are nonempty by definition, we can choose $v_0 \in V$ and define

$$g \colon W \to T, \quad w \mapsto \varphi(v_0, w).$$

We have to show that $g \circ \mathrm{pr}_2 = \varphi$. Take an open affine neighbourhood $U \subseteq T$ of $\varphi(v_0, w_0)$. Since $T \setminus U$ is closed in T, the inverse image $\varphi^{-1}(T \setminus U)$ is closed in $V \times W$ and by completeness of V, also $C := \mathrm{pr}_2(\varphi^{-1}(T \setminus U))$ is closed in W. By the definition of C, we have

$$W \setminus C = \{w \in W : \varphi(V \times \{w\}) \subseteq U\}.$$

Since $\varphi(V, w_0) = \{\varphi(v_0, w_0)\} \subseteq U$, we have $w_0 \in W \setminus C$. So $W \setminus C$ is a nonempty open subspace of the irreducible space W, hence dense in W. Since $V \times \{w\}$ is isomorphic to V, the variety $V \times \{w\}$ is complete and, as V is irreducible, also connected. Therefore, whenever $w \in W \setminus C$, the map φ restricts to a regular map $V \times \{w\} \to U$ from a complete connected variety to an affine variety, hence to a constant map (2.9(e)). Now

$$\varphi(V \times \{w\}) = \varphi(v_0, w) = g(w)$$

for all $w \in W \setminus C$. We can conclude that φ and $g \circ q$ agree on the dense subset $V \times (W \setminus C)$ of $V \times W$. Since the algebraic variety T is separated, φ and $g \circ q$ agree on $V \times W$. $\square$

Corollary 2.15. *Let V, W and T be algebraic varieties such that V is complete and irreducible and such that W irreducible. Let $\varphi \colon V \times W \to T$ be a regular map. Suppose that there are $v_0 \in V, w_0 \in W$ and $t_0 \in T$ such that*

$$\varphi(V \times \{w_0\}) = \{t_0\} = \varphi(\{v_0\} \times W).$$

Then $\varphi(V \times W) = \{t_0\}$.

Proof. By the Rigidity Theorem (2.14), there is $g \colon W \to T$ such that $\varphi(v, w) = g(w)$ for all $v \in V$ and $w \in W$. Then $\varphi(v, w) = g(w) = \varphi(v_0, w) = t_0$ for all $v \in V$ and all $w \in W$. $\square$

Intuitively, the previous corollary means that if two *coordinate axes* of $V \times W$ are collapsed to a point, then already $V \times W$ is collapsed to a point. In this sense, $V \times W$ is *rigid*.

2.3 Abelian varieties

Having recalled some algebraic geometry, we will now discuss Abelian varieties. Excellent resources on their theory are Milne's online notes [Mil08] and the book of Mumford [Mum70].

The best way to motivate them is to first look at one dimensional Abelian varieties, which are *elliptic curves*. For simplicity, let k be an algebraically closed field with $\mathrm{char}(k) \notin \{2, 3\}$. Good references for elliptic curves are the books by Silverman, e.g. [Sil09].

Definition 2.16. An **elliptic curve** E over k is a projective plane curve over k of the form

$$E : Y^2 = 4X^3 - aX - b$$

where $a, b \in k$ with $a^3 - 27b^2 \neq 0$.

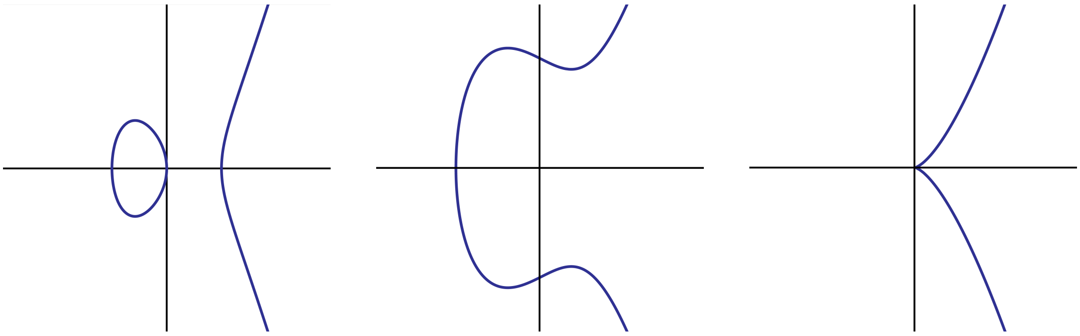
Remark. The equation above defines an *affine* curve in $\mathbb{A}^2(k)$. The *projective* in the definition means that we take the closure of this curve in $\mathbb{P}^2(k)$, i.e. the solution set of the homogenised equation

$$ZY^2 = 4X^3 - aXZ^2 - bZ^3$$

in $\mathbb{P}^2(k)$. Geometrically, taking this closure amounts to taking the affine curve and adding the point with homogeneous coordinates $[0 : 1 : 0]$ at infinity. The condition $a^3 - 27b^2 \neq 0$

guarantees that the curve is smooth. We will also call an algebraic variety that is isomorphic to a curve given as above an *elliptic curve*. In this sense, the definition above is a bit arbitrary and different authors have different conventions for the coefficients. Although we cannot easily visualise $\mathbb{P}^2(\mathbb{C})$, we can draw the real solutions to the affine equation of an elliptic curve.

Examples 2.17. Let $k = \mathbb{C}$ and consider the following curves as projective curves in $\mathbb{P}^2(\mathbb{C})$.

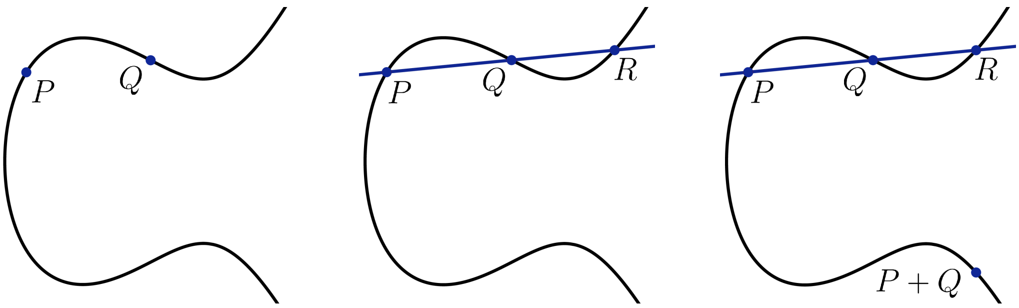


$E_1 \colon Y^2 = 4X^3 - X$ $\quad$ $E_2 \colon Y^2 = 4X^3 - X + 1$ $\quad$ $E_3 \colon Y^2 = 4X^3$

The curves E_1 and E_2 are elliptic curves, whereas E_3 is not an elliptic curve.

Elliptic curves carry a group structure, which can be defined in various ways. Let us elaborate on the very geometric *chord-and-tangent* construction.

Construction 2.18. Define the neutral element P_0 to be the point at infinity $[0 : 1 : 0]$, and set $P_0 + Q = Q + P_0 = Q$ for all Q on E. Define the sum of two *distinct* points P and Q on the affine part of E as follows: If P and Q have the same X-coordinate, define $P + Q$ to be P_0, the point at infinity. If P and Q have distinct X-coordinates, then take a line through P and Q. This line will intersect the affine part of E in precisely one other point R. Define $P + Q$ to be the point obtained from R by flipping the sign of the Y-coordinate.



Addition of two points on the affine part with distinct Y-coordinates.

Define the sum of a point P on the affine part *with itself* as follows: Take a tangent line at P. If this line does not intersect the affine part in a point different from P, define $P + P = P_0$. Otherwise, take the other affine intersection point, flip its Y-coordinate, and define the resulting point as $P + Q$. The fact that we indeed have the right intersection multiplicities in this construction follows from Bézout's theorem.

There are various proofs that this actually defines a commutative group structure on E, cf. III. Proposition 2.2. in [Sil09]. The inverse of a point on the affine part is obtained by flipping its Y-coordinate, and we have $-P_0 = P_0$. It can be proved that addition $+ \colon E \times E \to E$ and the inverse map $- \colon E \to E$ are regular maps, cf. Theorem 3.6. in [Sil09].

Abelian varieties will be natural generalisations of elliptic curves to higher dimensions. From the current very explicit definition, it is not clear how such a generalisation can be achieved.

In fact, there are no easy descriptions for Abelian varieties of dimension larger than 1 in terms of the homogeneous polynomials defining them. We need a different approach. There are two paths we can take here, the analytic and the algebraic one, whose connection is essential to the Pila-Zannier method. We will start with the algebraic approach and come back to the analytic approach later on (2.51). The key result for the algebraic one is the following:

Theorem 2.19. *Let V be a projective curve. The following are equivalent:*

(a) The curve V is isomorphic to an elliptic curve (as defined above).

(b) The curve V is a connected and admits a a group structure such that multiplication $V \times V \to V$ and the inverse map $V \to V$ are regular.

The proof requires more tools of algebraic geometry and topology than we can cover in our exposition (e.g. the Riemann-Roch theorem and the Lefschetz trace formula), cf. Proposition 3.1 in [Sil09] and the introduction of [Mil08].

The first step towards a definition Abelian varieties is the definition of algebaic groups.

Definition 2.20. An **algebraic group** over k is an algebraic variety G over k with a group structure defined by regular maps $m\colon G \times G \to G$ (multiplication) and $i\colon G \to G$ (inverse), as well as a distinguished point $e \in G$ (neutral element).

Example 2.21.

(a) Elliptic curves are algebraic groups.

(b) The n-th power $\mathbb{G}^n$ of multiplicative group $\mathbb{G} = \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$, which we studied in the first section, is an algebraic group.

(c) The general linear groups $\mathrm{GL}_n(k) := \{A \in \mathrm{Mat}_n(k) : \det(A) \neq 0\}$ and the special linear groups $\mathrm{SL}_n(k) := \{A \in \mathrm{Mat}_n(k) : \det(A) = 1\}$ are algebraic groups. They are not commutative for $n \geq 2$.

Definition 2.22. An **Abelian variety** over k is a connected and projective algebraic group over k. A **morphism** of Abelian varieties is a regular map that is also a group homomorphism. The **dimension** of an Abelian variety is its dimension as an algebraic variety.

The historically more correct definition of Abelian varieties would be to replace *projective* by *complete*. We already proved that every projective variety is complete (2.10). The projectivity of a complete connected algebraic group, however, is highly nontrivial (cf. Section 6 in [Mil08]).

Example 2.23.

(a) Elliptic curves are Abelian varieties. Also their products are Abelian varieties.

(b) An important example that motivated the study of Abelian varieties are *Jacobian varieties*, cf. part 3 in [Mil08].

(c) The algebraic groups $\mathbb{G}^n$ are *not* an Abelian varieties, because they fail to be complete. It suffices to check this for $\mathbb{G}^1 \cong \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$ (2.9(d)). This is an affine variety isomorphic to $V(XY - 1) \subseteq \mathbb{A}^2(\mathbb{C})$, hence a connected affine variety (we should not be fooled by the real picture). Since every complete connected affine variety is a point (2.9(e)), we conclude that $\mathbb{G}^1$ is not complete.

It is remarkable how short our definition of an ablian variety is. We do *not* assume that Abelian varieties are Abelian as groups, smooth or irreducible. It is surprising that all of these properties hold.

Commutativity is a direct consequence of the Rigidity Theorem (2.14):

Proposition 2.24. *The group structure of an Abelian variety is commutative.*

Proof. We consider the conjugation map

$$c\colon A \times A \to A, \quad (a,b) \mapsto aba^{-1}b^{-1}.$$

Because multiplication and inverses are regular, so also is c. But $c|_{\{e\}\times A}$ and $c|_{A\times\{e\}}$ are constant with image $\{e\}$. Hence also c is constant with image $\{e\}$ by a corollary of the Rigidity Theorem (2.15). We conclude that $ab = ba$ for all $a, b \in A$, so A is an Abelian group. $\square$

Because of the proved commutativity, we will write the group law of an Abelian variety additively from now on. A result of a similar flavour is the following:

Proposition 2.25. *Let A and B be Abelian varieties. Every regular map $\alpha\colon A \to B$ is the composition of a group homomorphism with a translation.*

Proof. Let $b := \alpha(0) \in B$. We can compose α with translation by $-b$ to assume that $\alpha(0) = 0$. It is then left to show that α is a group homomorphism. Consider the map

$$\varphi\colon A \times A \to B, \quad (a, a') \mapsto \alpha(a + a') - \alpha(a) - \alpha(a').$$

It is regular, as it is the difference of the two regular maps $m \circ (\alpha \times \alpha)\colon A \times A \to B \times B \to B$ and $\alpha \circ m\colon A \times A \to A \to B$. We have $\varphi(\{0\} \times A) = \{0\} = \varphi(A \times \{0\})$, hence $\varphi(A \times A) = \{0\}$ by rigidity (2.15). Thus, α is a group homomorphism. $\square$

In particular, every regular map $A \to B$ sending 0 to 0 is already a group homomorphism (hence a morphism of Abelian varieteis). We obtain the following corollary, which is also associated with the slogan *Geometry determines Algebra.*

Corollary 2.26. *It two Abelian varieties are isomorphic as varieties, then they are also isomorphic as groups.*

Proposition 2.27. *Abelian varieties are smooth and irreducible.*

Proof. Let A be an Abelian variety. Recall that the set of smooth points of any variety is dense and open (by Theorem 4.37 in [Mil17] this holds for every open affine, hence also for the entire variety). So there is a regular point $a \in A$.

Now let $a' \in A$ be any point. The morphism $t_{a'-a}\colon A \to A,\ b \mapsto b + a' - a$ is an isomorphsim of varieties sending a to a'. Hence also a' is regular. In more detail, let us use the definition that $b \in A$ is smooth if b lies on a single irreducible component C of A and the dimension of this component equals the dimension of the tangent space of A at b, cf. Definition 4.35 in [Mil17]. The automorphism $t_{a'-a}$ induces a bijection on the irreducible components of A, hence a' lies on a single irreducible component. It also induces an isomorphism on the tangent spaces $T_aA \to T_{a'}A$. Let C_a and $C_{a'}$ be the irreducible components on which a and a' lie, respectively, then

$$\dim C_{a'} = \dim C_a = \dim T_aA = T_{a'}A$$

and A is smooth at a'. Since a' was arbitrary, A is smooth. Moreover, a smooth connected variety is irreducible. $\square$

Definition 2.28. Let A be an Abelian variety. An **Abelian subvariety** of A is a closed algebraic subvariety of A that is an Abelian variety with the induced group structure.

A closed algebraic subgroup G of an Abelian variety A is not necessarily an Abelian subvariety, because it might fail to be connected. However, we can take the irreducible component (equivalently, the connected component) G° of G that contains the neutral element. Then G° is a connected closed subvariety of A and since the inclusion $G^\circ \to A$ is a regular map sending 0 to 0, it is already a group homomorphism (2.25). Thus G° is an Abelian subvariety of A.

Definition 2.29. Let A be an Abelian variety and G an algebraic subgroup of A. We call the Abelian subvariety G° of A the **identity component** of G.

2.4 Analytification and complex tori

The Zariski topology differs from the Euclidean topology in many ways. The Zariski topology is not Hausdorff, products of varieties do not carry the product topology and there are Zariski quasi-compact nonclosed sets. It is therefore surprising that in fact strong statements can be made on the connections between these topologies if the ground field is $\mathbb{C}$. One of the most important references on this topic is Serre's *Géométrie algébrique et géométrie analytique*, often just called GAGA, which is mainly concerned with the sheaf cohomology for both algebraic and analytic spaces. [Ser56]. A more elementary treatment of some of Serre's GAGA results can be found in [Wer11]. As a start, it is not hard to prove the following:

Proposition 2.30 (Essentially Theorem 3.1. in [Wer11])**.** *We can equip all algebraic varieties over $\mathbb{C}$ with a unique topology, which we will call the Euclidean topology, such that we have:*

(a) For $\mathbb{A}^n(\mathbb{C})$, the Euclidean topology is the Euclidean topology on $\mathbb{C}^n$.

(b) For closed subsets of $\mathbb{A}^n(\mathbb{C})$, i.e. algebraic sets, the Euclidean topology is the induced topology from the Euclidean topology on $\mathbb{A}^n(\mathbb{C})$.

(c) Regular maps of algebraic varieties are Euclidean continuous.

(d) The Euclidean topology is finer that the Zariski topology.

The construction yields a functor $-_{\mathrm{an}} \colon \mathsf{AlgVar}_{\mathbb{C}} \to \mathsf{Top}$ from the category of algebraic varieties to the category of topological spaces.

The Euclidean topology behaves as we would expect. For example, the Euclidean topology is Hausdorff, and the Euclidean topology on $X \times Y$ is the product topology of the Euclidean topologies on X and Y.

In the following eight results we collect facts that will be needed in the following discussion and the proof of Manin-Mumford. In all of them, varieties are assumed to be complex varieties.

Proposition 2.31. *Let V be an irreducible algebraic set of Krull dimension $n > 1$. Then V is unbounded in the Euclidean topology.*

Proof. By the geometric form of Noether normalisation (on p. 42 in [Mum88]), there is a surjective regular map $V \to \mathbb{A}^n(\mathbb{C})$. It becomes a continuous surjective map for the Euclidean topology. Since $\mathbb{C}^n$ is not Euclidean compact for $n > 1$, also V cannot be Euclidean compact. However, V is Euclidean closed, as the Euclidean topology is finer than the Zariski topology. Therefore, V must be unbounded in the Euclidean topology. □

Proposition 2.32 (I. §10, Thm 1. in [Mum88])**.** *Let V be an irreducible variety. If U is a nonempty Zariski open subvariety of V, then U is Euclidean dense in X.*

Recall that a subset of a variety V is *Zariski constructible* if it is a finite union of locally-closed subsets of V.

Corollary 2.33 (I. §10, Cor. 1. in [Mum88])**.** *Let V be an irreducible variety and let $Z \subseteq V$ be Zariski constructible. Then the Zariski closure and the Euclidean closure of Z in V agree.*

We have motivated completeness by considering it as the Zariski analogue for Euclidean compactness. By the following theorem, this is more than just an analogy.

Proposition 2.34 (I. §10, Thm 2. in [Mum88])**.** *Let V be an irreducible variety. Then V is complete if and only if V is Euclidean compact.*

Every Euclidean connected variety is also Zariski connected, because the former topology is finer than the latter. Surprisingly, we also have the following:

Proposition 2.35 (Thm 8.3. in [Wer11])**.** *Let V be an irreducible variety (in particular Zariski connected). Then V is Euclidean connected.*

In fact, even more is true. The following two results will not be used in this section, but will be relevant for minor details in the proof of Manin-Mumford. A proof of the second part in the following proposition is attributed to Ramanujam and can be found in an article of Ramanan [R+78].

Proposition 2.36. *Let V be an irreducible variety. Then V is Euclidean path connected. In fact, any two points on an irreducible variety can be joined by an irreducible curve.*

Proposition 2.37 (Corollary 4.16. in [Mum76])**.** *Let V be an irreducible complex projective variety and $W \subsetneq V$ a proper closed subvariety. Then $V \setminus W$ is Euclidean connected.*

The following proposition, however, is essential to the rest of this section.

Proposition 2.38 (Thm. 4.5. in [Wer11])**.** *If V is a smooth variety, then in the Euclidean topology, V can be given the structure of a complex analytic manifold. This can be realised in a functorial way. We have a commutative diagram of functors*

$$\begin{tikzcd} \mathsf{SmAlgVar}_{\mathbb{C}} \arrow[r, "-_{\mathrm{an}}"] \arrow[d, hook] & \mathsf{AnMan}_{\mathbb{C}} \arrow[d] \\ \mathsf{AlgVar}_{\mathbb{C}} \arrow[r, "-_{\mathrm{an}}"'] & \mathsf{Top}, \end{tikzcd}$$

where the functor on the right sends a manifold to its underlying space.

Now suppose that A is an Abelian variety over $\mathbb{C}$. Since A is Zariski irreducible (2.27), A is Euclidean connected (2.35). Since A is also complete (2.10), A is Euclidean compact (2.34). Because A is smooth (2.27), A can be given the structure of a smooth manifold (2.38), and by the functoriality of $-_{\mathrm{an}}\colon \mathsf{SmAlgVar}_{\mathbb{C}} \to \mathsf{AnMan}_{\mathbb{C}}$, the resulting manifold inherits a commutative group structure given by smooth maps. In conlclusion, this proves the following:

Proposition 2.39. *An Abelian variety over $\mathbb{C}$ is a compact and connected complex Lie group.*

Remark. Of course, we mean that the analytification A_{an} with the induced maps $+_{\mathrm{an}}$ and $-_{\mathrm{an}}$ is a compact connected complex Lie group. If the context is clear, we will just write $A, +$ and $-$. For a definition of complex Lie groups, cf. IV.2. in [FG02].

Remark. We did not include the commutativity in the statement (2.39), because every compact connected complex Lie group is commutative. This gives another proof for the commutativity of the group law of Abelian varieties over $\mathbb{C}$, cf. 1.(1) in [Mum70].

As a result, the rich analytic theory of complex Lie groups can be used to understand Abelian varieties. We will now investigate how exactly the Lie groups arising from Abelian varieties look like. Recall the following definition.

Definition 2.40. Let V be a finite dimensional complex vector space. A **lattice** in V is a discrete additive subgroup Λ of V. We say that a lattice $\Lambda \subseteq V$ is **full** if it has rank $\dim_{\mathbb{R}} V$.

Recall that for a lattice $\Lambda \subseteq V$ there exist $\mathbb{R}$-linearly independent $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_r$.

Definition 2.41. Let Λ be a lattice in V and let $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_r$. Then we call

$$\left\{ \sum_{j=1}^{r} a_j \lambda_j : a_j \in [0,1) \text{ for } 1 \leq j \leq r \right\}.$$

a **fundamental domain** for Λ.

Remark. A fundamental domain for Λ is in particular a fundamental domain for the natural action of Λ on V by translation, i.e. it contains precisely one point from each orbit.

The following two lemmas have very elementary proofs which we ommit.

Lemma 2.42. *Let V be a finite dimensional complex vector space and let $\Lambda \subseteq V$ be a subgroup. The following are equivalent.*

(a) The subgroup Λ is discrete, i.e. a lattice.

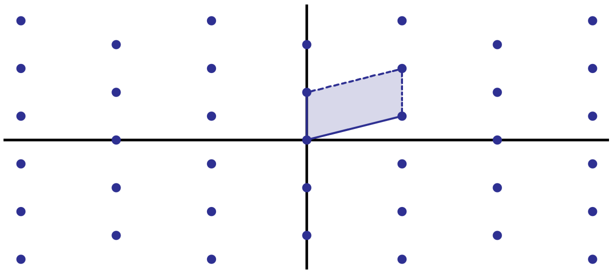
(b) There is an open subset U of V such that $U \cap \Lambda = \{0\}$.

Lemma 2.43. *Let V be a finite dimensional complex vector space and let $\Lambda \subseteq V$ be a lattice. The following are equivalent:*

(a) The lattice Λ has full rank.

(b) The (topological) quotient V/Λ by the translation action of Λ on V is compact.

Example 2.44. A full lattice in $\mathbb{C}$ and a fundamental domain (shaded) for it.



We will see that every Abelian variety is isomorphic as a complex Lie group to a *complex torus*, which is defined as follows:

Definition 2.45. A **complex torus** is a complex Lie group of the form V/Λ where V is a finite dimensional complex vector space and $\Lambda \subseteq V$ is a full lattice.

Remark. The quotient map $\pi\colon V \to V/\Lambda$ is always a covering map, because the group action is free and properly discontinuous. In fact, topologically, a complex torus V/Λ is indeed a torus $V/\Lambda \cong (S^1)^{\dim_{\mathbb{R}} V}$, and π is its universal covering. As complex manifolds, however, different lattices will yield different complex tori.

We will prove that every Abelian variety is a complex torus. For this, we need some differential geometry. The following proposition summarises some of the results in sections IV.5 and IV.6 in the book of Boothby [Boo86]. He considers real Lie groups, but the same proofs work in the complex case.

Proposition 2.46. *Let G be a complex Lie group with neutral element e. For every $v \in T_eG$, there is a unique analytic group homomorphism $\varphi_v\colon \mathbb{C} \to G$ with the following properties: We have $\varphi_v(0) = e$ and $d\varphi_v(0) = v$, the map $\varphi_-(-)\colon T_e(G) \times \mathbb{C} \to G$, $(v, z) \mapsto \varphi_v(z)$ is analytic and we have $\varphi_{zv}(w) = \varphi_v(zw)$ for all $v \in T_e(G)$ and all $z, w \in \mathbb{C}$.*

Consider the map $\pi\colon T_eG \to G$, $v \mapsto \varphi_v(1)$. Then π is analytic and we have $\varphi_v(z) = \pi(zv)$ for all $v \in V$ and $z \in \mathbb{C}$. Moreover, if G is commutative, then π is a group homomorphism. If we identify the tangent space at 0 of $T_e(G)$ with $T_e(G)$ itself, then the differential of π at 0 is the identity $T_e(G) \to T_e(G)$.

Definition 2.47. Let G be a complex Lie group with neutral element e. We call φ_v the **one-parameter-subgroup** of v and π the **exponential map** for G.

Example 2.48. Consider $\mathbb{G} = \mathbb{C}^\times$ as a complex Lie group. The tangent space of $\mathbb{G}$ at 1 is (isomorphic to) $\mathbb{C}$ and π is the usual exponential map $z \mapsto e^z$.

We can now describe the exponential map for Abelian varieties.

Proposition 2.49. *Let A be an Abelian variety of dimension g over $\mathbb{C}$. Let*

$$\pi \colon T_0(A) \to A$$

be the exponential map for the complex Lie group A. Then π is surjective and its kernel is a full lattice in $T_0(A)$. In particular, choosing a basis for $T_0(A)$, we obtain an isomorphism of complex Lie groups $A \cong \mathbb{C}^g/\Lambda$ for some full lattice $\Lambda \subseteq \mathbb{C}^g$.

Remark. By $T_0(A)$, we mean the analytic tangent space. We should remark that the Zariski and manifold tangent spaces of a smooth variety at point will always be isomorphic. For algebraic sets in $\mathbb{C}^n$, this can be seen because both tangent spaces can be defined by the same linear polynomials, cf. section 4.b in [Mil17].

Proof of Proposition 2.49, following 2.1. in [Mil08]. Let us first show that π is surjective. Since π is a homomorphism, its image H in A is certainly a subgroup of A. The differential $d\pi$ of π is an isomorphism, as it is the identity under a suitable choice of a basis (2.46). By the inverse mapping theorem, π restricts to a biholomorphic map on a neighbourhood of 0. In particular, there is an open neighbourhood U of 0 in H. Then since H is additively closed, for every $a \in H$ we get an open neighbourhood $a + U$ of a contained in H, so H is open. But

$$A \setminus H = \bigcup_{a \in A \setminus H} a + H$$

is the union of translates of H, hence open too. So H is nonempty and both open and closed in A, thus $H = A$ because A is connected. This proves the surjectivity of π. Since π is biholomorphic between neighbourhoods of the respective neutral elements, there is a neighbourhood U of 0 in $T_0(A)$ such that $\pi|_U$ is injective. Phrased differently,

$$U \cap \ker(\pi) = \ker(\pi|_U) = \{0\}.$$

But this suffices to see that $\ker(\pi)$ is a lattice in $T_0(A)$ (2.42). Furthermore, the quotient $T_0(A)/\ker(\pi) \cong \operatorname{im}(\pi) = A$ is compact (2.39), so $\ker(\pi)$ is a full lattice in T_0A (2.43). □

Corollary 2.50. *Let A be an Abelian variety of dimension g over $\mathbb{C}$. The torsion points A_{tor} in A are isomorphic to $(\mathbb{Q}/\mathbb{Z})^{2g}$ as a group. Moreover, for $n \in \mathbb{N}$, the torsion points $A[n]$ in A of order n are isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ as a group.*

Proof. As a complex Lie group, so in particular as a group, A is isomorphic to $\mathbb{C}^g/\Lambda$ for some full lattice $\Lambda \subseteq \mathbb{C}^g$. Then

$$A_{\text{tor}} \cong \mathbb{Q}\Lambda/\Lambda \cong (\mathbb{Q}/\mathbb{Z})^{2g} \quad \text{and} \quad A[n] \cong \frac{1}{n}\Lambda/\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^{2g},$$

because $\Lambda \cong \mathbb{Z}^{2g}$. □

We have seen that every Abelian variety of dimension g over $\mathbb{C}$ yields a full lattice $\Lambda \subseteq \mathbb{C}^g$ such that $A \cong \mathbb{C}^g/\Lambda$ as complex Lie groups. It is a natural question whether the converse holds, i.e. if for every full lattice $\Lambda \subseteq \mathbb{C}^g$ there is is an Abelian variety A over $\mathbb{C}$ of dimension g such that $A \cong \mathbb{C}^g/\Lambda$. For elliptic curves, this is true and we can give a very explicit construction.

Proposition 2.51 (VI.3 Prop. 3.6 in [Sil09]). *Let $\Lambda \subseteq \mathbb{C}$ be a full lattice. The coefficients*

$$g_2 := 60 \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{\lambda^4} \quad \text{and} \quad g_3 := 140 \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{\lambda^6}$$

are well-defined and $g_2^3 - 27g_3^3$ is nonzero. Thus $E\colon Y^2 = 4X^3 - g_2 X - g_3$ is an elliptic curve. The Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

is indeed a well-defined function and the map

$$\pi\colon \mathbb{C} \to E \subseteq \mathbb{P}^2(\mathbb{C}), \quad z \mapsto \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{for } z \in \mathbb{C} \setminus \Lambda \\ [0 : 1 : 0] & \text{for } z \in \Lambda \end{cases}$$

is a homomorphism of complex Lie groups with kernel Λ.

So indeed, every complex torus $\mathbb{C}/\Lambda$ is isomorphic as a complex Lie group to the analytification of an elliptic curve. This is in fact an equivalent definition of elliptic curves (by the equivalence of categories below (2.55)). However, the same cannot be said about higher dimensional Abelian varieties. In general, a complex torus $\mathbb{C}^g/\Lambda$ is isomorphic to the analytification of an Abelian variety of dimension g if and only if the lattice Λ admits a *Riemann form*:

Definition 2.52. Let V be a finite dimensional complex vector space and let Λ be a full lattice in V. A **Riemann form** on (V, Λ) is a symplectic bilinear form $s\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that for the $\mathbb{R}$-bilinear extension $s_{\mathbb{R}}\colon V \times V \to \mathbb{R}$ of s, the following hold:

(a) For all $u, v \in V$, we have $s_{\mathbb{R}}(iu, iv) = s_{\mathbb{R}}(u, v)$.

(b) For all $v \in V \setminus \{0\}$, we have $s_{\mathbb{R}}(iv, v) > 0$.

A complex torus V/Λ is **polarisable** if (V, Λ) admits a Riemann form.

It can be shown that Riemann forms $s\colon \Lambda \times \Lambda \to \mathbb{Z}$ on (V, Λ) on Λ are in correspondence with positive dimenional Hermitian forms $h\colon V \times V \to \mathbb{C}$ such that $h(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. Using this correspondence and with substantially more work, the following theorem can be proved.

Theorem 2.53 (Section 3, Corollary in [Mum70]). *A complex torus V/Λ is isomorphic to the analytification of an Abelian variety if and only if V/Λ is polarisable.*

Definition 2.54. Let V/Λ and V'/Λ' be complex tori. A **morphism of complex tori** is a $\mathbb{C}$-linear map $\Phi\colon V \to V'$ such that $\Phi(\Lambda) \subseteq \Lambda'$.

If A and A' are complex Abelian varieties isomorphic to tori V/Λ and V'/Λ', respectively, then every regular map $\varphi\colon A \to A'$ lifts to a linear map $\Phi\colon V \to V'$ such that $\Phi(\Lambda) \subseteq \Lambda'$. On the other hand, every linear map $\Phi\colon V \to V$ with $\Phi(\Lambda) \subseteq \Lambda'$ induces a holomorphic map $V/\Lambda \to V/\Lambda'$. It then follows from Chow's theorem (cf. chapter 4 in [Mum76]) applied to the graph of this holomorphic map, that it is in fact regular. We obtain:

Theorem 2.55. *The analytification functor $-_{\mathrm{an}}$ restricts to an equivalence between the category of Abelian varieties to the category of polarisable tori.*

This theorem is remarkable because it gives us an equivalence between an *algebraic* and an *analytic* category (although these terms are not strict). The equivalence can be exploited for many important constructions such as the *dual Abelian variety* and, with more work via line bundles, *quotients of Abelian varieties by Abelian subvarieties*. We will have to work with them in the proof of Manin-Mumford, so let us state what we need.

Proposition 2.56. *Let A be an Abelian variety and let $B \subseteq A$ be an Abelian subvariety. Then there exists an Abelian variety C and a surjective morphism of Abelian varieties $\varphi\colon A \to C$ such that* $\ker(\varphi) = B$*. In particular, the underlying group of C is isomorphic to the quotient group A/B. If $\pi\colon \mathbb{C}^g \to A$ is the exponential map for A and $\Theta \subseteq \mathbb{C}^g$ is a complex vector subspace of A such that $\pi(\Theta) = B$, then we get an induced map of groups $\mathbb{C}^g/\Theta \to A/B$ which is the exponential map for C for an identification of $\mathbb{C}^g/\Theta$ with $T_0(C)$.*

Definition 2.57. In the situation of the previous proposition, we call C the **quotient Abelian variety** of A by B and denote it by A/B.

Quotients of Abelian varieties by subvarieties are not contained in Mumfords book [Mum70] or Milne's notes [Mil08]. For the construction, we refer to [Lin14].

3 O-minimality and the Pila-Wilkie Counting Theorem

This section is devoted to the model theoretic background of the Pila-Zannier method. After an introductory subsection concerning examples of o-minimal structures and the *Cell Decomposition Theorem*, we discuss the *Pila-Wilkie Counting Theorem*, which allows o-minimality to enter the world of Diophantine geometry.

Throughout, let $\mathcal{R} = (R, <, \ldots)$ be a model of the theory of dense linear orders without endpoints. By *definable* we will always mean *definable in* $\mathcal{R}$. For $a \in R$ the sets $[a, \infty)$, (a, ∞), $(-\infty, a]$, $(-\infty, a)$ and R itself are understood to be intervals in R. In this section, we always equip R with the topology induced by the order, and R^m with the respective product topology.

3.1 Basic o-minimal geometry

Let us start with presenting some results on o-minimal structures, mostly without proofs. An excellent introduction to o-minimal geometry is the book of van den Dries [vdD98].

Definition 3.1. The structure $\mathcal{R}$ is **o-minimal** if every definable subset of R is a finite union of intervals and points.

Although this condition looks quite simple, proving the o-minimality of a structure can be a very hard task. It often requires a lot of effort studying the algebra, analysis and model theory involved. Let us elaborate on that by giving some examples.

Examples 3.2.

(a) The *real ordered field* $\mathbb{R}_{\text{alg}} = (\mathbb{R}, +, -, \cdot, 0, 1, <)$ is o-minimal. This follows from quantifier elimination for the theory of real closed fields, which is also known as the Tarski-Seidenberg theorem. Geometrically, it asserts that a coordinate projection of a semialgebraic set is again semialgebraic. In fact, the study of o-minimality was strongly motivated by the aim to generalise results from semialgebraic geometry, cf. [Tar49, vdD86].

(b) The *real ordered field with restricted analytic functions* $\mathbb{R}_{\text{an}}$ is o-minimal. This is implicitly due to Łojasievicz and Gabrielov, cf. [Łoj65] and [Gab68], and was extracted within the framework of o-minimality by van den Dries, cf. [vdD86]. The structure $\mathbb{R}_{\text{an}}$ will be central for our proof of Manin-Mumford, so let us make its definition precise. We call a tuple $r = (r_1, \ldots, r_m) \in \mathbb{R}^m_{>0}$ a *polyradius*. For polyradii $r, s \in \mathbb{R}^m_{>}$, we write $r > s$ if $r_j > s_j$ for all $j \in \{1, \ldots, m\}$. We define $\mathbb{R}\{X_1, \ldots, X_m\}_r$ to be $\left\{\sum_{\alpha \in \mathbb{N}^m} b_\alpha X^\alpha \in \mathbb{R}[\![X_1, \ldots, X_m]\!] : \sum_{\alpha \in \mathbb{N}^m} |b_\alpha| r^\alpha < \infty\right\}$ and set

$$\mathbb{R}\{X_1, \ldots, X_m\}_{1^+} := \bigcup_{r > (1, \ldots, 1)} \mathbb{R}\{X_1, \ldots, X_m\}_r,$$

the power series that converge in some neighbourhood of the cube $[-1, 1]^m$. Every formal power series $f \in \mathbb{R}\{X_1, \ldots, X_m\}_{1^+}$ defines a function

$$\tilde{f} \colon \mathbb{R}^m \to \mathbb{R}, \quad \tilde{f}(x) := \begin{cases} f(x) & \text{for } x \in [-1, 1]^m \\ 0 & \text{for } x \in \mathbb{R}^m \setminus [-1, 1]^m. \end{cases}$$

We call a function of this form a *restricted analytic function*. We define $\mathcal{L}_{\text{an}}$ to be the language of ordered rings $\{+, -, \cdot, <, 0, 1\}$ augmented by new function symbols for each restricted analytic function $\tilde{f} \colon \mathbb{R}^m \to \mathbb{R}$ for all $m \in \mathbb{N}$. The *real ordered field with restricted analytic functions*, denoted by $\mathbb{R}_{\text{an}}$, is the $\mathcal{L}_{\text{an}}$-structure $(\mathbb{R}, 0, 1, +, -, \cdot, (\tilde{f}_i)_i)$, the $\tilde{f}_i$ being all interpreted naturally as restricted analytic functions. Of course, the cube $[-1, 1]^m$ can be replaced by its image under any definable automorphism of $\mathbb{R}^m$.

(c) The *real ordered field with the exponential function* $\mathbb{R}_{\exp} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$ is o-minimal. This was proved by Wilkie, who more generally studied the model theory of the real ordered field expanded by *pfaffian functions*, cf. [Wil96].

(d) Merging the two previous examples, also the *real ordered field with restricted analytic functions and (unrestricted) exponentiation* $\mathbb{R}_{\text{an,exp}} = (\mathbb{R}, +, -, \cdot, 0, 1, <, (\tilde{f}_i), \exp)$ is o-minimal. This was first proved by Miller and van den Dries [vdDM94] and reproved by van den Dries, Mackintyire and Marker, who axiomatised the theory of $\mathbb{R}_{\text{an,exp}}$ [vdDMM94]. Both proofs require a lot of algebra. The structure $\mathbb{R}_{\text{an,exp}}$ is the structure most used in Diophantine applications. In particular, the recent proof of the André-Oort conjecture requires the o-minimality of $\mathbb{R}_{\text{an,exp}}$ and not just of $\mathbb{R}_{\text{an}}$ [PST21].

The o-minimality of these structures would not have been studied so intensively if it was not for the striking consequences of o-minimality. Arguably, the *Cell Decomposition Theorem* is the most fundamental result among these consequences. According to it, the innocent looking one-dimensional condition characterising o-minimal structures also forces the definable sets in higher dimensions to obey a tame structure theory. Following van den Dries [vdD98], let us introduce the relevant definitions and state the theorem.

Definition 3.3. For $X \subseteq R^m$ definable in $\mathcal{R}$, we set

$$C(X) := \{f : X \to R : f \text{ is definable and continuous}\},$$
$$C_\infty(X) := C(X) \cup \{-\infty, \infty\},$$

viewing $-\infty$ and ∞ as constant functions on X. For $f, g \in C_\infty(X)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$. Let $(i_1, \ldots, i_m) \in \{0,1\}^m$. With induction on m, we define an $\boldsymbol{(i_1, \ldots, i_m)}$**-cell** in R^m as follows:

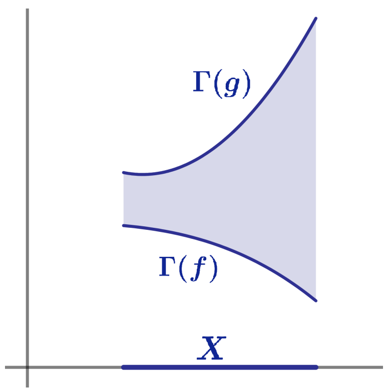
(a) A (0)-cell is a singleton $\{r\} \subseteq R$. A (1)-cell is an interval $(a, b) \subseteq R$.

(b) Supoose that $(i_1, \ldots, i_m)$-cells are defined. An $(i_1, \ldots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where X is an $(i_1, \ldots, i_m)$-cell. An $(i_1, \ldots, i_m, 1)$-cell is a set of the form

$$\{(x, r) \in X \times R : f(x) < r < g(x)\}$$

for an $(i_1, \ldots, i_m)$-cell X and $f, g \in C_\infty(X)$. with $f < g$.

We also define a $()$-cell to be the one point space R^0. A **cell** in R^m is an $(i_1, \ldots, i_m)$-cell for some $(i_1, \ldots, i_m) \in \{0,1\}^m$. We call the $(1, \ldots, 1)$-cells and the unqiue $()$-cell **open cells**. The **dimension** of an $(i_1, \ldots, i_m)$-cell is the natural number $i_1 + \cdots + i_m$ and the dimension of the $()$-cell is $-\infty$.



Visualisation of cells.

The open cells are indeed open in R^m and all other cells have empty interior in R^m.

Lemma 3.4. *Let $i = (i_1, \dots, i_m) \in \{0,1\}^m$. For every i-cell $C \subseteq R^m$ there is a coordinate projection $\mathrm{pr}_i \colon R^m \to R^k$, only depending on i, that maps C homeomorphically to an open cell. We can choose $k = i_1 + \dots + i_m$.*

Proof. Let $j(1) < \dots < j(k)$ be precisely the indices such that $i_{j(\ell)} = 1$ for $1 \le \ell \le k$. Then we have $k = i_1 + \dots + i_m$. Define

$$\mathrm{pr}_i \colon R^m \to R^k, \quad (x_1, \dots, x_m) \mapsto (x_{j(1)}, \dots, x_{j(k)}).$$

It is not hard to prove by induction on m that this restricts to a homeomorphism from each i-cell to an open cell. □

Definition 3.5. A **decomposition** of R^m is a finite partition $\mathcal{D} = \{C_1, \dots, C_n\}$ of R^m into cells that is defined inductively on m as follows.

(a) A decomposition of R^1 is a finite partition of R^1 of the form

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty), \{a_1\}, \dots, \{a_k\})\},$$

for some $a_1, \dots, a_k \in R$.

(b) Suppose that it is defined what a decomposition of R^m is. A decomposition of R^{m+1} is a finite partition of R^{m+1} into cells $\{C_1, \dots, C_n\}$ such that the set $\{\mathrm{pr}(C_i) : i \in \{1, \dots, k\}\}$ is a decomposition of R^m, where $\mathrm{pr} \colon R^{m+1} = R^m \times R \to R^m$ denotes the projection.

Let $X \subseteq R^m$. We say that a decomposition $\mathcal{D}$ of R^m **partitions** X if for each cell $C \in \mathcal{D}$ we have $C \subseteq X$ or $C \cap X = \emptyset$. Then we call $\{C \in \mathcal{D} : C \subseteq X\}$ a **cell decomposition** for X.

Each definable set admits a cell decomposition, and in fact even more holds. The statement of the *Cell Decomposition Theorem* is as follows:

Theorem 3.6 (Cell Decomposition Theorem, Ch. 3, Thm. 2.11. in [vdD98])**.** *Let $m \in \mathbb{N}$.*

(a) Let $X_1, \dots, X_n \subseteq R^m$ be definable. Then there is a decomposition of R^m partitioning each of the $X_1, \dots, X_n$.

(b) Let $X \subseteq R^m$ and let $f \colon X \to R$ be a definable function. Then there is a decomposition $\mathcal{D}$ of R^m partitioning X such that for each $C \in \mathcal{D}$ with $C \subseteq X$ the restriction $f|_C \colon C \to R$ is continuous.

The proof of this theorem is by a nested induction and takes several pages. For $m = 1$, the first part of the theorem is the definition o-minimality and the second statement is the *monotonicity theorem* for o-minimal structures, cf. 1.2. in [vdD98].

Having introduced the dimension of a cell and knowing that each definable set has a cell decomposition, it is natural to define the dimension of a definable set X to be the maximal dimension of a cell in a cell decomposition for X. This does not depend on the chosen decomposition, which is an immediate consequene of the following result.

Proposition 3.7 (c.f. 3.14, 3.17(4) in [Cos99] and 3.20 in [Cos02])**.** *Let $X \subseteq R^m$ be definable and nonempty. Then the following natural numbers are well-defined and coincide:*

(a) the maximal $d \in \mathbb{N}$ such that there exists an injective map $R^d \to X$,

(b) the maximal $d \in \mathbb{N}$ such that there is a d-dimensional cell in a cell decomposition for X.

For $\mathcal{R} = \mathbb{R}_{\mathrm{alg}}$ and $X \subseteq \mathbb{R}^m$, these numbers also agree with the Krull dimension of the real Zariski closure of X in $\mathbb{R}^m$.

Definition 3.8. Let $X \subseteq R^m$ be definable. The **dimension** of X, denoted by $\dim(X)$, is maximal $d \in \mathbb{N}$ such that there is a d-dimensional cell in some cell decomposition for X if $X \neq \emptyset$ and $-\infty$ if $X = \emptyset$.

It can be proved that the dimension behaves as expected. For example, for a definable map $f\colon X \to R^n$, we have $\dim(f(X)) \leq \dim(X)$ and for definable $X \subseteq R^m$ and $Y \subseteq R^n$, we have $\dim(X \times Y) = \dim(X) + \dim(Y)$. Among many other results, this fits into the general conception that o-minimality provides a *tame* structure theory, not allowing pathological phenomena such as space filling curves.

Another interesting consequence of the Cell Decomposition Theorem is that every definable set only has finitely many *definably connected components*, which are by definition the maximal subsets of X that are not the disjoint union of two definable nonempty open subsets. These components of a definable set also partition it, cf. Ch. 3 Proposition 2.18 in [vdD98].

We finish this subsection with a brief discussion of *definable families*, which will be relevant for the proof sketch of Pila-Wilkie in the next part. The presentation of the following results is based on [BvdD22].

Definition 3.9. Suppose that $S \subseteq \mathbb{R}^d$ and $\mathcal{X} \subseteq S \times \mathbb{R}^m$ are definable. For $s \in S$ we define

$$\mathcal{X}(s) := \{x \in \mathbb{R}^m : (s, x) \in X\}.$$

We call $(\mathcal{X}(s))_{s \in S}$ a **definable family**.

Example 3.10. Let $e \in \mathbb{N}_{\geq 1}$. Recall that a *hypersurface* in $\mathbb{R}^m$ is a real algebraic set that is the zero set $V(f)$ of a single nonzero polynomial $f \in \mathbb{R}[X_1, \dots, X_m]$. We say that a hypersurface $H = V(f)$ has *degree* at most e if f has degree at most e. For $m \in \mathbb{N}_{\geq 1}$, the hypersurfaces in $\mathbb{R}^m$ of degree at most e form a semialgebraic family. Indeed, the dimension of the real vector space $\mathfrak{P}$ of polynomials in m variables of degree at most e is $d := \binom{e+m}{m}$. Let $\Phi\colon \mathbb{R}^d \to \mathfrak{P}$ be an isomorphism of real vector spaces, $A := \mathbb{R}^d \setminus \{0\}$ and define

$$\mathcal{H} := \{(a_1, \dots, a_d, b_1, \dots, b_m) \in \mathbb{R}^d \times \mathbb{R}^m : \mathbb{R}_{\text{alg}} \models a \neq 0 \wedge \Phi(a_1, \dots, a_d)(b_1, \dots, b_m) = 0\}.$$

Then both $\mathcal{H}$ and A are definable in $\mathbb{R}_{\text{alg}}$, each $\mathcal{H}(a) = V(\Phi(a))$ for $a \in A$ is a hypersurface of degree at most m and every such hypersuface is of this form.

Proposition 3.11 (Essentially Ch. 3, Cor. 3.6. [vdD98])**.** *Let $(\mathcal{X}(s))_{s \in S}$ be a definable family given by definable $S \subseteq R^d$ and $\mathcal{X} \subseteq S \times R^m$. Let $\mathcal{D}$ be a decomposition of R^{d+m} partitioning $\mathcal{X}$. Let* pr *denote the projection $R^{d+m} \to R^d$, $(x_1, \dots, x_{d+m}) \mapsto (x_1, \dots, x_d)$. Then for all $s \in S$, the set*

$$\mathcal{D}(s) := \{C(s) : C \in \mathcal{D}, s \in \text{pr}(C)\}$$

is a decomposition of R^m partitioning $\mathcal{X}(s)$. Here, as for $\mathcal{X}(s)$, we write $C(s)$ for the set $\{x \in R^m : (x, s) \in C\}$. In particular, there is $M \in \mathbb{N}$ such that each $\mathcal{X}(s)$ has at most M definably connected components.

From the Cell Decomposition Theorem (3.6), the previous proposition (3.11) and our preceding discussion of hypersurfaces (3.12), the following can be deduced:

Corollary 3.12. *Let $m, e \in \mathbb{N}_{\geq 1}$ and let $d := \binom{e+m}{m}$. As defined above (3.10), let $\mathcal{H}$ be the subset of $\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^m$ such that $(\mathcal{H}(a))_{a \in \mathbb{R}^d \setminus \{0\}}$ is the semialgebraic family of hypersurfaces in $\mathbb{R}^m$ of degree at most e. Then there are $L \in \mathbb{N}_{\geq 1}$ and semialgebraic subsets $\mathcal{C}_1, \dots, \mathcal{C}_L$ of $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^m$ such that the following holds. For all $a \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\mathcal{H}(a) = \bigcup_{\ell=1}^{L} \mathcal{C}_\ell(a)$$

and for each $\ell \in \{1, \dots, L\}$ there is a tuple $i^\ell = (i_1^\ell, \dots, i_m^\ell) \in \{0,1\}^m \setminus \{(1, \dots, 1)\}$ such that each $\mathcal{C}_\ell(a)$ is a semialgebraic i^ℓ-cell or empty.

3.2 The Pila-Wilkie Counting Theorem

This section is devoted to the *Pila-Wilkie Counting Theorem*. A complete proof of this theorem would be too lengthy, whereas the mere statement of it would not do justice to its central and characterising role in the Pila-Zannier strategy. Therefore, we will provide some context, state main ingredients and deduce the theorem from them. This subsection is based on the first two sections of an expository paper by Bhardwaj and van den Dries [BvdD22].

Recall the definition of the height function $\mathsf{H}\colon \mathbb{Q}^m \to \mathbb{N}_{\geq 1}$ (1.3), and that for $X \subseteq \mathbb{R}^m$ we set

$$X(\mathbb{Q}) := X \cap \mathbb{Q}^m, \quad X(\mathbb{Q}, t) := \{x \in X(\mathbb{Q}) : \mathsf{H}(x) \leq t\}, \quad \text{and} \quad \mathsf{N}(X, t) := \#X(\mathbb{Q}, t).$$

Furthermore, recall the definition of the algebraic part X^{alg} and the trascendental part X^{trans} of $X \subseteq \mathbb{R}^m$ (1.5). The most common form of the Pila-Wilkie theorem goes as follows:

Theorem 3.13 (Thm. 1.8. in [PW06])**.** *Let X be definable in an o-minimal expansion of $\mathbb{R}_{\text{alg}}$. Then for all $\varepsilon > 0$ there exists a constant C, only depending on X and ε, such that $\mathsf{N}(X^{\text{trans}}, t) \leq Ct^{\varepsilon}$ for all $t \in \mathbb{R}_{\geq 0}$.*

We will discuss a slightly stronger version that gives a uniform bound for definable families.

Theorem 3.14 (Thm. 1.9. in [PW06])**.** *Let $(\mathcal{X}(s))_{s \in S}$ be a definable family in an o-minimal expansion of $\mathbb{R}_{\text{alg}}$ given by definable $S \subseteq \mathbb{R}^d$ and $\mathcal{X} \subseteq S \times \mathbb{R}^m$. Then for all $\varepsilon > 0$ there exists a constant C, only depending on $\mathcal{X}$ and ε, such that $\mathsf{N}(\mathcal{X}(s)^{\text{trans}}, t) \leq Ct^{\varepsilon}$ for all $t \in \mathbb{R}_{\geq 0}$ and all $s \in S$.*

We should remark that it is possible to strengthen the theorem even further, cf. Theorem 1.10 in [PW06]. Towards a proof, we start with three short lemmas that provide a better understanding of the algebraic and transcendental parts.

Lemma 3.15. *Let $m \in \mathbb{N}_{\geq 1}$ and let $X \subseteq \mathbb{R}^m$ be open. Then $X^{\text{trans}} = \emptyset$.*

Proof. For every $x \in X$ there is $\varepsilon > 0$ such that the infinite connected semialgebraic set $(x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_m - \varepsilon, x_m + \varepsilon)$ is a subset of X containing x, so $x \in X^{\text{alg}}$. □

Lemma 3.16. *Suppose that $X = X_1 \cup \cdots \cup X_n$. Then $X^{\text{trans}} \subseteq X_1^{\text{trans}} \cup \cdots \cup X_n^{\text{trans}}$. More precisely, if $x \in X_i \cap X^{\text{trans}}$, then $x \in X_i^{\text{trans}}$.*

Proof. Let $x \in X^{\text{trans}}$ and $k \in \{1, \ldots, n\}$ such that $x \in X_i$. By the definition of the algebraic part, we have $X_1^{\text{alg}} \cup \cdots \cup X_m^{\text{alg}} \subseteq X^{\text{alg}}$, hence $x \notin X_i^{\text{alg}}$, and we conclude that $x \in X_i^{\text{trans}}$. □

Lemma 3.17. *Let $W \subseteq \mathbb{R}^m$ be semialgebraic and $X \subseteq W$. Let $f\colon W \to \mathbb{R}^{\ell}$ be an injective semialgebraic map. If $f|_X\colon X \to f(X)$ is a homeomorphism, then $f(X^{\text{alg}}) = f(X)^{\text{alg}}$ and $f(X^{\text{trans}}) = f(X)^{\text{trans}}$.*

Proof. Since $f|_X\colon X \to f(X)$ is bijective, it suffices to prove $f(X^{\text{alg}}) = f(X)^{\text{alg}}$. Let S be an infinite connected semialgebraic subset of X. Since f is injective and semialgebraic, $f(S)$ is infinite and semialgebraic. Since $f|_X$ is continuous, $f(S)$ is connected. This proves that $f(X^{\text{alg}}) \subseteq f(X)^{\text{alg}}$. Now let T be an infinite connected semialgebraic subset of $f(X)$. Since f and T are semialgebraic, $f^{-1}(T) \subseteq W$ is semialgebraic. Since f is injective, $f^{-1}(T) \subseteq f^{-1}(f(X)) = X$. Hence, because $f|_X$ is a homeomorphism, $f^{-1}(T) \subseteq X$ is also infinite and connected. This proves $f(X^{\text{alg}}) \supseteq f(X)^{\text{alg}}$, and we conclude that $f(X^{\text{alg}}) = f(X)^{\text{alg}}$. □

In the proof of Pila-Wilkie below, we will just refer to these lemmas as the *Interior Lemma*, the *Union Lemma* and the *Function Lemma*, respectively.

As announced, we will use two nontrivial intermediate theorems to deduce the Counting Theorem. The first result does not concern o-minimality. According to it, the rational points on a set can be covered effectively by hypersurfaces if the set admits a *strong k-parametrisation*. Let us define this term and make the statement precise.

Definition 3.18. Let $X \subseteq \mathbb{R}^m$ and let $k \in \mathbb{N}_{\geq 1}$. A **strong k-parametrisation** of X is a C^k-map $f\colon (0,1)^n \to \mathbb{R}^m$ with $n < m$ and image X such that $|f^{(\alpha)}(b)| \leq 1$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ and all $b \in (0,1)^n$. Here, $f^{(\alpha)}$ denotes the derivative $(\partial^{|\alpha|}/\partial x^\alpha)f$.

In the following, we write $x = x(y,z)$ to indicate that x only depends on y and z.

Theorem 3.19 (Thm. 1.2. in [BvdD22])**.** *Let $m \geq 1$. Then for all $e \geq 1$ there are $k = k(m,e) \in \mathbb{N}_{\geq 1}$, $\varepsilon = \varepsilon(m,e) \in \mathbb{R}_{>0}$, and $c = c(m,e) \in \mathbb{R}_{>0}$ such that if $X \subseteq \mathbb{R}^m$ has a strong k-parametrisation. Then there exists $N \in \mathbb{N}$ with $N \leq ct^\varepsilon$ such that N hypersurfaces in $\mathbb{R}^m$ of degree at most e suffice to cover $X(\mathbb{Q},t)$. We can arrange that $\varepsilon(m,e) \to 0$ for $e \to \infty$.*

In the proof of Pila-Wilkie, we will just refer to this theorem as the *Hypersurface Theorem*. It is not clear why rational points lying on a hypersurface should be easier to control. The key motivation for considering hypersurfaces in the present proof is that they decrease the dimension, thereby allowing for an induction argument.

The second main ingredient guarantees the existence of strong k-parametrisations under suitable assumptions, and will enable us to apply the Hypersurface Theorem (3.19) in the proof of Pila-Wilkie.

Theorem 3.20 (Thm. 1.2. in [BvdD22])**.** *Let $\mathcal{R}$ be an o-minimal extension of $\mathbb{R}_{\text{alg}}$. Let $(\mathcal{X}(s))_{s\in S}$ be a definable family in $\mathcal{R}$ given by definable $S \subseteq \mathbb{R}^\ell$ and $\mathcal{X} \subseteq S \times \mathbb{R}^m$. Assume that for all $s \in S$, the set $\mathcal{X}(s)$ is contained in $[-1,1]^m$ and has empty interior in $\mathbb{R}^m$. Then for every $k \in \mathbb{N}_{\geq 1}$ there is $M \in \mathbb{N}$ such that each $\mathcal{X}(s)$ is a union of at most M subsets, each of which admits a strong k-parametrisation.*

Below, we will refer to this theroem as the *Parametrisation Theorem*. In contrast to the *Hypersurface Theorem* (3.19), the proof of the Paramtetrisation Theorem requires some model theory, such as passing to $\aleph_0$-saturated elementary extensions.

Proof of Theorem 3.14, following section 2 of [BvdD22]. Let $(\mathcal{X}(s))_{s\in S}$ be definable family of subsets $\mathcal{X}(s) \subseteq \mathbb{R}^m$ in an o-minimal expansion of $\mathbb{R}_{\text{alg}}$ and let $\varepsilon > 0$. Our goal is to find a constant C such that

$$\mathsf{N}(\mathcal{X}(s)^{\text{trans}}, t) = \#\mathcal{X}(s)^{\text{trans}}(\mathbb{Q}, t) \leq Ct^\varepsilon$$

for all $t \in \mathbb{R}_{\geq 0}$. Since $\mathsf{N}(\mathcal{X}(s)^{\text{trans}}, t)$ is zero for $t < 1$, we can assume throughout that $t \geq 1$. We proceed by induction on m. For $m = 0$, each $\mathcal{X}(s)$ is empty or a singleton, so we can take $C := 1$. Now let $m > 0$ and suppose that the claim holds for all $m' < m$. To apply the Parametrisation Theorem (3.20) to X, we need the following reduction.

Claim 1. We can assume that $\mathcal{X}(s)$ is contained in $[-1,1]^m$ and has empty interior in $\mathbb{R}^m$ for all $s \in S$.

We fix $s \in S$ and consider $X := \mathcal{X}(s)$. Let us deal with the first condition. For any $Z \subseteq \mathbb{R}^m$ and $I \subseteq \{1,\dots,m\}$ we consider the set

$$Z_I := \{z \in Z : |z_i| > 1 \text{ for all } i \in I \text{ and } |z_i| \leq 1 \text{ for all } i \notin I\}.$$

We also consider the map

$$f_I\colon \mathbb{R}^m_I \to \mathbb{R}^m, \quad (a_1,\dots,a_m) \mapsto (b_1,\dots,b_m),$$

with $b_i := 1/a_i$ for $i \in I$ and $b_i := a_i$ for $i \notin I$. The map f_I is a homeomorphism onto its image $f(\mathbb{R}^m_I)$, which is contained in $[-1,1]^m$. Moreover, for $q \in \mathbb{Q}^m \cap \mathbb{R}^m_I$ we have $\mathsf{H}(q) = \mathsf{H}(f_I(q))$ since inverting rational numbers does not change their height. Both the set $\mathbb{R}_I$ and the map $f_I\colon \mathbb{R}^m_I \to \mathbb{R}^m$ are semialgebraic. Furthermore, f_I is injective and maps each $X_I \subseteq \mathbb{R}_I$ homeomorphically onto its image. Thus, we can conclude by the Function Lemma (3.17) that

$$f_I(X_I^{\text{trans}}) = f_I(X_I)^{\text{trans}}.$$

By these observations, we have $\mathsf{N}(X_I^{\mathrm{trans}}, t) = \mathsf{N}(f_I(X_I)^{\mathrm{trans}}, t)$ for each I. We can write each X as a disjoint union $X = \bigcup_{I \subseteq \{1,\dots,m\}} X_I$, so by the Union Lemma (3.16) we have $X^{\mathrm{trans}} \subseteq \bigcup_{I \subseteq \{1,\dots,m\}} X_I^{\mathrm{trans}}$. We can conclude that

$$\mathsf{N}(X^{\mathrm{trans}}, t) \leq \sum_{I \subseteq \{1,\dots,m\}} \mathsf{N}(X_I^{\mathrm{trans}}, t) = \sum_{I \subseteq \{1,\dots,m\}} \mathsf{N}(f_I(X_I)^{\mathrm{trans}}, t).$$

Since $f_I(X_I) \subseteq [-1,1]^m$ and the number of summands only depends on m, it indeed suffices to consider subsets of $[-1,1]^m$. We can also assume that X has empty interior in $[-1,1]^m$, which can be seen as follows. Let X° be the interior of X in $\mathbb{R}^m$. Then $X = (X) \cup (X \setminus X^\circ)$, and thus $X^{\mathrm{trans}} \subseteq (X^\circ)^{\mathrm{trans}} \cup (X \setminus X^\circ)^{\mathrm{trans}}$ by the Union Lemma (3.16). However, $(X^\circ)^{\mathrm{trans}} = \emptyset$ by the Interior Lemma (3.15), so we can indeed remove the interior of X in $\mathbb{R}^m$ from X. $\square$(Claim 1)

Let us come back to the main proof. We choose $e \in \mathbb{N}_{\geq 1}$ sufficiently large such that for $k = k(m,e) \geq 1$, $\varepsilon(m,e)$ and $c = c(m,e)$ as in the Hypersurface Theorem (3.19), we have $\varepsilon(m,e) \leq \varepsilon/2$, with respect to the $\varepsilon > 0$ we are given. By the Parametrisation Theorem (3.20), we find $M \in \mathbb{N}$ such that each $\mathcal{X}(s)$ is a union

$$\mathcal{X}(s) = \bigcup_{i=1}^{M} Y(s)_i$$

of at most M subsets, each of which admits a strong k-paramtetrisation. By the Hypersurface Theorem (3.19), we can cover each $Y(s)_i(\mathbb{Q}, t)$ by at at most $ct^{\varepsilon(m,e)}$ hypersurfaces in $\mathbb{R}^m$ of degree at most e. Let $N \coloneqq \lfloor ct^{\varepsilon(m,e)} \rfloor$ and let $H_{i1}(s), \dots, H_{iN}(s)$ be these hypersurfaces, so

$$\mathcal{X}(s)(\mathbb{Q}, t) = \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} (\mathcal{X}(s)(\mathbb{Q}, t) \cap H_{ij}(s)).$$

By the Union Lemma (3.16), for $x \in \mathcal{X}(s)^{\mathrm{trans}} \cap H_{ij}(s)$, we have $x \in (\mathcal{X}(s)^{\mathrm{trans}} \cap H_{ij}(s))^{\mathrm{trans}}$, so we can conclude that

$$X(s)^{\mathrm{trans}}(\mathbb{Q}, t) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} (\mathcal{X}(s) \cap H_{ij}(s))^{\mathrm{trans}}(\mathbb{Q}, t).$$

We will now estimate $\mathsf{N}((\mathcal{X}(s) \cap H_{ij}(s))^{\mathrm{trans}}, t)$ from above. We want the bound to be independent of the chosen hypersurface and s.

Claim 2. There exists a constant $c' \in \mathbb{R}_{>0}$ such that for any hypersurface H in $\mathbb{R}^m$ of degree at most e and all $s \in S$, we have $\mathsf{N}((\mathcal{X}(s) \cap H)^{\mathrm{trans}}, t) \leq c' t^{\varepsilon/2}$ for all $t \in \mathbb{R}_{\geq 0}$.

Let $d \coloneqq \binom{m+e}{m}$ be the dimension of the real vector space of polynomials in m variables of degree up to e. We discussed at the end of the previous subsection that there are definable families $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^m$ for some $L \in \mathbb{N}_{\geq 1}$ such that $(\mathcal{H}(a))_{a \in \mathbb{R}^d \setminus \{0\}}$ is the collection of hypersurfaces in $\mathbb{R}^m$ of degree at most e, we have

$$\mathcal{H}(a) = \bigcup_{\ell=1}^{L} \mathcal{C}_\ell(a)$$

for all $a \in \mathbb{R}^d \setminus \{0\}$, and for each $\ell \in \{1, \dots, L\}$ there is $i^\ell = (i_1^\ell, \dots, i_m^\ell) \in \{0,1\}^m \setminus \{(1, \dots, 1)\}$ such that each $\mathcal{C}_\ell(a)$ is a semialgebraic i^ℓ-cell or empty (3.12). By the Union Lemma (3.16), we have

$$(\mathcal{X}(s) \cap \mathcal{H}(a))^{\mathrm{trans}} \subseteq \bigcup_{\ell=1}^{L} (\mathcal{X}(s) \cap \mathcal{C}_\ell(a))^{\mathrm{trans}} \tag{1}$$

Let $\mathrm{pr}_{i^\ell} \colon \mathbb{R}^m \to \mathbb{R}^{m_\ell}$ be the coordinate projection that maps each $C_\ell(a)$ homeomorphically to an open cell or to the empty set if $C_\ell(a) = \emptyset$ (3.4). Then each $\mathcal{X}(s) \cap \mathcal{C}_\ell(a)$ is mapped homeomorphically by pr_{i^ℓ} to its image

$$\mathcal{X}'_\ell(s,a) := \mathrm{pr}_{i^\ell}(\mathcal{X}(s) \cap \mathcal{C}_\ell(a)) \subseteq \mathrm{pr}_{i^\ell}(C_\ell(a)) \subseteq \mathbb{R}^{m_\ell}.$$

If $q \in \mathcal{C}_\ell(a) \cap \mathbb{Q}^m$, then $\mathrm{pr}_{i^\ell}(q) \in \mathbb{Q}^{m_\ell}$ and $\mathsf{H}(\mathrm{pr}_{i^\ell}(q)) \leq \mathsf{H}(q)$, because we take the maximum of the heights over fewer coordinates. Furthermore, by the Function Lemma (3.17), we have $\mathcal{X}'_\ell(s,a)^{\mathrm{trans}} = \mathrm{pr}_{i^\ell}((\mathcal{X}(s) \cap \mathcal{C}_\ell(a)^{\mathrm{trans}})$. Hence we get

$$\mathsf{N}((\mathcal{X}(s) \cap \mathcal{C}_\ell(a))^{\mathrm{trans}}, t) \leq \mathsf{N}((\mathcal{X}'_\ell(s,a)^{\mathrm{trans}}, t). \tag{2}$$

We can consider the $\mathcal{X}'_\ell(s,a)$ as a members of a definable family $(\mathcal{X}'_\ell(s,a))_{(s,a) \in S \times (\mathbb{R}^d \setminus \{0\})}$ with

$$\mathcal{X}'_\ell = \{(s,a,x) \in (S \times (\mathbb{R}^d \setminus \{0\})) \times \mathbb{R}^{m_\ell} : x \in \mathrm{pr}_{i^\ell}(\mathcal{X}(s) \cap \mathcal{C}_\ell(a))\}.$$

We have $m_\ell = i_1^\ell + \cdots + i_m^\ell < m$, so by the induction hypothesis, for all $\ell \in \{1, \ldots, L\}$ we find constants $c_\ell > 0$ with

$$\mathsf{N}(\mathcal{X}'_\ell(s,a)^{\mathrm{trans}}, t) \leq c_\ell t^{\varepsilon/2} \tag{3}$$

Let us conclude the claim. If H is a hypersurface in $\mathbb{R}^m$ of degree at most e, then there is $a \in \mathbb{R}^d \setminus \{0\}$ such that $H = \mathcal{H}(a)$. Given any $s \in S$, for all $t \in \mathbb{R}_{\geq 0}$ we obtain

$$\begin{aligned}
\mathsf{N}((\mathcal{X}(s) \cap \mathcal{H}(a))^{\mathrm{trans}}, t) &\overset{(1)}{\leq} \sum_{\ell=1}^{L} \mathsf{N}((\mathcal{X}(s) \cap \mathcal{C}_\ell(a))^{\mathrm{trans}}, t) \\
&\overset{(2)}{\leq} \sum_{\ell=1}^{L} \mathsf{N}((\mathcal{X}'_\ell(s,a)^{\mathrm{trans}}, t) \\
&\overset{(3)}{\leq} L(c_1 + \cdots + c_\ell) t^{\varepsilon/2}.
\end{aligned}$$

The desired statement follows with $c' := L(c_1 + \cdots + c_n)$. □(Claim 2)

Recall that we had already proved

$$\mathcal{X}(s)^{\mathrm{trans}}(\mathbb{Q}, t) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} (\mathcal{X}(s) \cap H_{ij}(s))^{\mathrm{trans}}(\mathbb{Q}, t).$$

By Claim 2., we also have $\#(\mathcal{X}(s) \cap H_{ij}(s))^{\mathrm{trans}}(\mathbb{Q}, t) \leq c' t^{\varepsilon/2}$ for all $t \in \mathbb{R}_{\geq 0}$ with c' independent of s, i and j. Reminding ourselves that we chose N such that $N \leq c t^{\varepsilon(m,e)}$, and e such that $\varepsilon(m,e) \leq \varepsilon/2$, we conclude that

$$\mathsf{N}(\mathcal{X}(s)^{\mathrm{trans}}, t) \leq M N c' t^{\varepsilon/2} \leq M c' c t^{\varepsilon/2} t^{\varepsilon/2} = M c' c t^{\varepsilon}.$$

The estimate in the Counting Theorem holds with $C := M c' c$. □

It is a natural question whether we can achieve sharper bounds. Allowing arbitrary definable sets in $\mathbb{R}_{\mathrm{an}}$, there are examples showing that the bound of the Counting Theorem cannot be improved (e.g. Pila's Example 7.5. in [Pil04]). The following result, known as *Wilkie's conjecture*, is Conjecture 1.11 in the paper in which Pila and Wilkie proved the Counting Theorem [PW06].

Theorem 3.21. *Let $X \subseteq \mathbb{R}^m$ be definable in $\mathbb{R}_{\exp}$. Then there are constants C and K depending only on X such that for $t \geq 1$ we have $N(X^{\mathrm{trans}}, t) \leq C(\log t)^K$.*

This result was proved in 2022 by Binyamini, Novikov and Zack [BNZ22], demonstrating that not only applications of the Pila-Wilkie theorem, but also variations of the theorem itself are of high interest in current research projects.

4 An o-minimality proof of Manin-Mumford

In this section, we will finally give a proof of Manin-Mumford. Recall the statement:

Theorem 4.1. *Let A be a complex Abelian variety and let $V \subseteq A$ be a closed algebraic subvariety. Suppose that V does not contain any translate of a positive dimensional Abelian subvariety of A. Then V contains only finitely many torsion points of A.*

We begin our journey towards a proof with a relation between algebraic, semialgebraic and analytic sets. In a second subsection, which forms the main part of the proof, we demonstrate the *Ax-Lindemann-Weierstraß theorem*. This can be recognised as the study of the *algebraic part* in the general pattern of the Pila-Zannier strategy and corresponds to the main Theorem 2.1. in the original paper [PZ08]. By proving Ax-Lindemann-Weierstraß, we do not strictly follow the original work of Pila and Zannier for two reasons. Firstly, the success of the Pila-Zannier strategy has led to some clarifications of arguments in the proof. These are not only preferable for simplicity, but also because they can be adapted more easily towards proofs of related problems such as the André-Oort conjecture. Secondly, we can (and will) give an o-minimality proof of Ax-Lindemann-Weierstraß. In Pila's and Zannier's original work, it is just a side note after the proof of Lemma 2.2. that their study of the algebraic part can be related to the o-minimality of $\mathbb{R}_{\text{an}}$. We will elaborate on this, emphasising the power of o-minimality in Diophantine applications. In a final part, we deduce Manin-Mumford. As in the general pattern, there will be an *algebraic part*, which corresponds to a well structured set of torsion points, and a *transcendental part*, which corresponds to finitely many torsion points. The *Ax-Lindemann-Weierstraß theorem* will help us to understand the algebraic part. With the *Pila-Wilkie Counting Theorem*, we will then see that there are *few* points on the trancendental part. As in the case of Mann's Theorem, we need to consider *Galois orbits* to get from few to finite. However, the required bound, given by *Masser's Theorem*, is much harder to obtain in the case of Abelian varieties. Its proof is of a completely different nature than the algebro- and tame-geometric considerations for the other parts. Therefore, as Pila and Zannier do, we will use Masser's result without a proof.

Many results obtained in this section require a lot of machinery from algebraic, analytic and semialgebraic geometry. While the previous two sections were aiming to give an introduction to the relevant concepts from algebraic geometry and o-minimality, other topics such as real algebraic varieties and analytic varieties remained untouched. The theory of *real algebraic varieties* and their interplay with complex algebraic varieties is very well explained by Mangolte [Man20]. For an overview on *analytic varieties* that covers most of the results we use in this section, we refer to Adamus [Ada13]. A detailed treatment of complex geometry is provided by the book of Fritzsche and Grauert [FG02].

4.1 Relating algebraic, semialgebraic and analytic sets

A central theme in the Pila-Zannier strategy is the study of connections between different kinds of geometry. This subsection is devoted to a result of this kind. The following proposition relates complex algebraic, (real) semialgebraic and complex analytic sets. It will play a key role for the proof of Ax-Lindemann-Weierstraß, as well as for the conclusion of Manin-Mumford.

Proposition 4.2. *Let Z be a complex analytic subset of $\mathbb{C}^g$ and let $W \subseteq \mathbb{C}^g$ be a connected irreducible semialgebraic subset such that $W \subseteq Z$. Then there exists an irreducible complex algebraic set X' in $\mathbb{C}^g$ such that $W \subseteq X' \subseteq Z$.*

The proof can be found below and is very technical. Nevertheless, the elegance of the statement connecting algebraic, semialgebraic and analytic geometry is remarkable for certain. The proposition generalises Lemma 2.1. from the original paper, in which Pila and Zannier only consider curves. The statement above (4.2) was first proved by Pila and Tsimerman (Lemma 4.1. in [PT13]). Before we give a proof, we have to define some terms in the statement and give a brief review on complex analytic sets.

As in the discussion of Mann's Theorem, by a semialgebraic subset of $\mathbb{C}^g$, we mean a subset of $\mathbb{C}^g$ that is mapped to a semialgebraic set in $\mathbb{R}^{2g}$ under any isomorphism of real vector spaces $\mathbb{C}^g \cong \mathbb{R}^{2g}$. We also define the following:

Definition 4.3. A semialgebraic set $X \subseteq \mathbb{R}^{2g}$ is **irreducible** if it is irreducible with the subspace topology from the real Zariski topology on $\mathbb{R}^{2g}$.

Remark. In other words, a semialgebraic set $X \subseteq \mathbb{R}^{2g}$ is irreducible if and only if it is nonempty and can not be written as a union $X = X' \cup X''$ where X' and X'' are proper closed subsets of X with respect to the subspace topology on from the real Zariski topology on $\mathbb{R}^{2g}$. As closures and dense subspaces of irreducible spaces are irreducible, X is irreducible if and only if its real Zariski closure in $\mathbb{R}^{2g}$ an irreducible real algebraic set.

We have devoted two sections to algebraic and o-minimal (hence also semialgebraic) geometry. Let us also summarise some definitions and results on *complex analytic sets*. For the rest of this subsection, by a *(sub)manifold* we always mean a *complex analytic (sub)manifold*.

In contrast to algebraic and semialgebraic sets, analytic sets are usually defined already by a local property:

Definition 4.4. Let M be manifold. We call a subset $Z \subseteq M$ **analytic** in M if for every $x \in M$ there exist an open neighbourhood U of x in M and holomorphic functions $f_1, \ldots, f_r \colon U \to \mathbb{C}$ such that $Z \cap U = \{z \in M : f_j(z) = 0 \textit{ for all } j \in \{1, \ldots, r\}\}$. We call Z **globally analytic** if we can arrange $U = M$.

Example 4.5. Every algebraic set is an analytic set in $\mathbb{C}^g$. Every projective set in $\mathbb{P}^g(\mathbb{C})$ is an analytic set in the analytification of $\mathbb{P}^g(\mathbb{C})$, i.e. in $\mathbb{P}^g(\mathbb{C})$ considered as a complex manifold.

Remark. An analytic set Z of a manifold M is closed in M, for if U^x denotes the neighbourhood of x in M for every $x \in M$ as in the definition, then we have $M \setminus Z = \bigcup_{x \in M} U^x \setminus (U^x \cap Z)$.

Analytic subsets behave like algebraic varieties in many ways. This could be made very clear sheaf-theoretically via the notion of analytic spaces and Serre's GAGA theorems, cf. section 9 in [Ada13] and [Ser56]. However, in the proof of Manin-Mumford we will in fact only encounter analytic sets in $\mathbb{C}^g$, so we will treat analytic sets hands-on.

It can be readily checked that analytic sets allow for many natural constructions. For example, preimages of analytic sets under holomorphic maps of manifolds are analytic. Also arbitrary unions and finite intersections of analytic subsets of a manifold are again analytic. In fact, also the following nontrivial result holds:

Proposition 4.6 (Thm. 7.11. in [Ada13])**.** *An arbitrary intersection of analytic subsets of a complex manifold M is again an analytic subset of M.*

There are natural notions for *dimension*, *smoothness*, and *irreducibility* for analytic sets:

Definition 4.7. Let M be a manifold and let $X \subseteq M$ be any subset. We define the **dimension of X** as

$$\dim(X) := \max\{\dim N : \ N \text{ is a submanifold of } M \text{ and } N \subseteq X\}$$

if $X \neq \emptyset$ and as $\dim(X) := -\infty$ if $X = \emptyset$. For $x \in M$, the **dimension of X at x** is

$$\dim_x(X) := \min\{\dim(X \cap U) : U \text{ us an open neighbourhood of } x \text{ in } M\}.$$

For elementary facts on these notions of dimension, cf. section 1.3. in [Ada13].

Definition 4.8. Let Z be an analytic subset of a manifold M. A point $z \in Z$ is **regular** or **smooth**, if there is an open neighbourhood U of z in M such that $Z \cap U$ is a submanifold of M. We denote the set of regular points of Z by Z^{reg}, and its complement, the set of **singular** points of Z, by Z^{sing}.

Proposition 4.9 (Thm. 2.11. and Thm. 9.16. in [Ada13])**.** *Let Z be an analytic subset of a manifold M. Then Z^{reg} is Euclidean open and dense in Z. On the other hand, Z^{sing} is again an analytic subset of M.*

Definition 4.10. Let Z be an analytic subset of a complex manifold M and let $p \in Z^{\text{reg}}$. The **tangent space** of Z at p, denoted by T_pZ, is the usual manifold tangent space $T_p(Z \cap U)$, where U is an open neighbourhood of p in M such that $Z \cap U$ is a submanifold of M.

Definition 4.11. An analytic subset Z of a manifold M is **(analytically) irreducible** if there are no proper subsets Z_1 and Z_2 of Z that are analytic subsets of M such that $Z = Z_1 \cup Z_2$.

Proposition 4.12 (Cor. 7.10 in [Ada13])**.** *Let Z be an irreducible analytic subset of a manifold M. If Y is an analytic subset of M with $Y \subseteq Z$ and $\dim Y = \dim Z$, then $Y = Z$.*

Every algebraic set is also an analytic set, and the introduced analytic notions behave well with the respective algebraic notions (and often they agree). This can usually be deduced from Serre's GAGA theorems, cf. [Ser56]. Although Serre‘s results are highly nontrivial, such comparison arguments are often considered standard in the literature. Comparison of tangent spaces, which will be important in the proof, can be justified for algebraic sets as follows:

The *Zariski* regular and the *analytically* regular points on an algebraic set coincide. This is the underlying reason why we get a functor $-_{\text{an}}\colon \mathsf{SmAlgVar}_{\mathbb{C}} \to \mathsf{AnMan}_{\mathbb{C}}$ and can be proved by quite elementary methods, cf. Theorem 4.5. in [Wer11]. Furthermore, also the Zariski tangent space and the analytic tangent space of an algebraic set at a smooth point agree. In $\mathbb{C}^g$, this holds as the respective naive constructions use the vanishing sets of the same linear polynomials: It does not matter whether we take *formal derivatives* for Zariski tangent spaces or *usual derivatives* for tangent spaces of submanifolds, cf. Section 4 in [Mil17].

Let us explain one more important result on the interplay of analytic and algebraic notions:

Proposition 4.13. *An algebraic set V in $\mathbb{C}^g$ is analytically irreducible if and only if it is algebraically irreducible.*

Proof. Certainly, if Z is analytically irreducible, then it is also algebraically irreducible. The other implication is far from trivial. An analytic set is irreducible if and only if its set of regular points is Euclidean connected (Corollary 7.9. in [Ada13]). But the *Zariski* regular and the *analytically* regular points of an algebraic set coincide, so it suffices to prove that the Zariski regular points V^{reg} of an irreducible algebraic set V are Euclidean connected. This is indeed the case: Corollary 4.16 in [Mum70] asserts that if X is an irreducible projective variety and $Y \subseteq X$ a proper closed subset, then $X \setminus Y$ is Euclidean connected. To use this in the present situation, recall that the singular points V^{sing} of an algebraic set $V \subseteq \mathbb{C}^g$ form a proper algebraic subset (Theorem 4.37 in [Mil17]). We can consider V as a subset of $\mathbb{P}^g(\mathbb{C})$ via $\mathbb{A}^g(\mathbb{C}) \cong U_0 \subseteq \mathbb{P}^g(\mathbb{C})$ (2.6). Let X be the closure of V in $\mathbb{P}^g(\mathbb{C})$, let $H := \mathbb{P}^g(\mathbb{C}) \setminus U_0 \cong \mathbb{P}^{g-1}(\mathbb{C})$ be the hyperplane at infinity and set $Y := (X \cap H) \cup V^{\text{sing}}$. Then Y is a proper closed subset of the irreducible projective variety X, and thus $X \setminus Y \cong V^{\text{reg}}$ is Euclidean connected. $\square$

With these facts at hand, let us prove the main result of this subsection (4.2).

Proof of Proposition 4.2, following 8.1. in [Orr15]. We are given a complex analytic set Z in $\mathbb{C}^g$ and a connected irreducible semialgebraic set $W \subseteq \mathbb{C}^g$ such that $W \subseteq Z$. We have to show that there is an irreducible complex algebraic set X' with $W \subseteq X' \subseteq Z$.

In this proof, it is convenient to identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$ by separating real and imaginary parts, so let $\Phi\colon \mathbb{R}^{2g} \to \mathbb{C}^g$ be the isomorphism of real vector spaces

$$(x_1, y_1, \ldots, x_g, y_g) \mapsto (x_1 + iy_1, \ldots, x_g + iy_g)$$

and let $X \subseteq \mathbb{R}^{2g}$ be a connected irreducible semialgebraic set such that $\Phi(X) = W$.

We will enlarge $\Phi(X)$ to obtain a complex algebraic set, and then verify that this set is irreducible and still contained in Z. Let us first take the real Zariski closure of X in $\mathbb{R}^{2g}$ and denote it by S. This is a real algebraic set in $\mathbb{R}^{2g}$, which we can make into a complex algebraic set $S_{\mathbb{C}}$ by defining $S_{\mathbb{C}}$ to be the complex Zariski closure $\mathrm{cl}^{\mathbb{C}\text{-Zar}}(\iota(S))$ of $\iota(S)$ in $\mathbb{C}^{2g}$. In fact, $S_{\mathbb{C}}$ is the *extension of scalars* of S, so it is defined by the same polynomials as S, cf. sections 2.1-2.3 in [Man20]. If we compose ι with the surjective regular map

$$f\colon \mathbb{C}^{2g} \to \mathbb{C}^g, \quad (x_1, y_1, \ldots, x_g, y_g) \mapsto (x_1 + iy_1, \ldots, x_g + iy_g),$$

then the resulting map $f \circ \iota\colon \mathbb{R}^{2g} \to \mathbb{C}^g$ is precisely the isomorphism Φ we use to identify $\mathbb{R}^{2g}$ with $\mathbb{C}^g$. Now let $S' \coloneqq f(S_{\mathbb{C}})$. The present situation is depicted in the following diagram:

$$\begin{tikzcd}
S \arrow[r, phantom, "\subseteq"] & \mathbb{R}^{2g} \arrow[d, hook, "\iota"] \arrow[dd, bend left=60, "\Phi", "\cong"'] \\
S_{\mathbb{C}} \arrow[r, phantom, "\subseteq"] & \mathbb{C}^{2g} \arrow[d, two heads, "f"] \\
S' \arrow[r, phantom, "\subseteq"] & \mathbb{C}^g
\end{tikzcd}$$

Finally, we define X' to be the complex Zariski closure of S' in $\mathbb{C}^g$. This finishes the construction of X'. We will now show that X' is a witness for the statement of the theorem. Our construction immediately yields

$$\Phi(X) \subseteq \Phi(S) = f(\iota(S)) \subseteq f(S_{\mathbb{C}}) = S' \subseteq X'.$$

It remains to prove the irreducibility of X' and that $X' \subseteq Z$.

Claim 1. We have $\dim_{\mathrm{Cell}} X = \dim_{\mathrm{Krull}} S$ and the complex algebraic set $S_{\mathbb{C}}$ is irreducible. In particular, also X' is irreducible.

Proof of Claim 1. It is a general result from semialgebraic geometry that the dimension of a semialgebraic set equals the Krull dimension of its real Zariski closure (e.g. Theorem 3.20 in [Cos02]). This proves the first part. For the second part, we first observe that S is irreducible by the definition of irreducibility for the semialgebraic set X. By separating real and imaginary parts, we see that the inclusion $\iota\colon \mathbb{R}^{2g} \to \mathbb{C}^g$ is continuous for the real and complex Zariski topologies, respectively. But then $S_{\mathbb{C}} = \mathrm{cl}^{\mathbb{C}\text{-Zar}}(\iota(S))$ is the closure of the image of an irrducible space under a continuous map, hence irreducible. We can conclude that X' is irreducible by the same arguments. Indeed, f is complex Zariski continuous as a regular map, so $S' = f(S_{\mathbb{C}})$ is irreducible. Again, because irreducibility is preserved under taking closures, $X' = \mathrm{cl}^{\mathbb{C}\text{-Zar}}(S')$ is irreducible. $\square$(Claim 1)

Claim 2. The set X' is also the Euclidean closure of S' in $\mathbb{C}^g$.

Proof of Claim 2. We defined X' to be the Zariski closure of $S' = f(S_{\mathbb{C}})$ in $\mathbb{C}^g$. The map $f\colon \mathbb{C}^{2g} \to \mathbb{C}^g$ is given by a polynomial in each coordinate, so it is a regular map. Since $S_{\mathbb{C}}$ is an algebraic set, it is in particular Zariski constructible and hence so also is $S' = f(S_{\mathbb{C}})$ by Cheavalley's theorem (Corollary 3.2.8. in [Mar02]). The Zariski closure and the Euclidean closure of subsets of Zariski constructable subsets of irreducible varieties agree (2.33). Since $\mathbb{C}^g$ is irreducible, the claim follows. □(Claim 2)

Claim 3. There exists no analytic set $\tilde{Z} \subseteq \mathbb{C}^g$ such that $\iota(X) \subseteq \tilde{Z} \subsetneq S_{\mathbb{C}}$.

Proof of Claim 3. Towards a contradiction, we assume that there is such an analytic set. Since analytic sets are stable under arbitrary intersections (4.6), we may pick $\tilde{Z}$ inclusion-minimal such that $\iota(X) \subseteq \tilde{Z} \subsetneq S_{\mathbb{C}}$. The singular points $\tilde{Z}^{\text{sing}}$ of an analytic set $\tilde{Z}$ in $\mathbb{C}^g$ form another analytic set in $\mathbb{C}^g$ (4.9). By the minimality of $\tilde{Z}$, we can thus conclude that $\iota(X) \not\subseteq \tilde{Z}^{\text{sing}}$, so $\iota(X) \cap \tilde{Z}^{\text{reg}} \neq \emptyset$. On the other hand, the smooth points $\tilde{Z}^{\text{reg}}$ of the analytic set $\tilde{Z}$ form a nonempty open subset of $\tilde{Z}$ (4.9). Therefore, the set

$$\iota|_X^{-1}(\tilde{Z}^{\text{reg}} \cap \iota(X))$$

of points $x \in X$ such that $\iota(x)$ is a smooth point of $\tilde{Z}$ is a nonempty open subset of X. In particular, it intersects a semialgebraic cell of maximal dimension in X nontrivially (3.3). We can thus pick $x \in X$ lying in a maximal dimensional semialgebraic cell of X such that $\tilde{Z}$ is smooth at $\iota(x)$. Since x lies on a maximal dimensional cell, also S is smooth at x, and S and X coincide in a Euclidean open neighbourhood of x.

We can conclude the proof with a comparision of tangent spaces. Since $\iota(X) \subseteq \tilde{Z}$, we have

$$\iota(T_x S) \subseteq T_{\iota(x)} \tilde{Z}.$$

Here, S is a real algebraic subset of $\mathbb{R}^{2g}$ and we consider its real Zariski tangent space $T_x S$ at the smooth point x, whereas the space $T_{\iota(x)}\tilde{Z}$ is the complex analytic tangent space of $\tilde{Z}$ at a smooth point $\iota(x)$. The map $\iota\colon \mathbb{R}^{2g} \to \mathbb{C}^{2g}$ is an $\mathbb{R}$-linear map of vector spaces and it follows that also the complex linear hull $\mathbb{C}\iota(T_x S)$ of $\iota(T_x S)$ is contained in the complex vector space $T_{\iota(x)}\tilde{Z}$. Moreover, unfolding the definitions, we see that

$$\mathbb{C}\iota(T_x S) = T_{\iota(x)} S_{\mathbb{C}}$$

because both spaces are the vanishing sets of the same linear polynomials. As a result, we get an inclusion of complex vector spaces

$$T_{\iota(x)} S_{\mathbb{C}} \subseteq T_{\iota(x)} \tilde{Z}.$$

Furthermore, we have $\dim T_{\iota(x)}\tilde{Z} = \dim_{\iota(x)} \tilde{Z}$, because $\tilde{Z}$ is smooth at $\iota(x)$. We also have $\dim_{\iota(x)} \tilde{Z} \leq \dim S_{\mathbb{C}}$, since $\tilde{Z} \subseteq S_{\mathbb{C}}$ by assumption. Since in general for Zariski tangent spaces we have $\dim S_{\mathbb{C}} \leq \dim T_{\iota(x)} S_{\mathbb{C}}$, we can conclude that

$$\dim T_{\iota(x)} S_{\mathbb{C}} \leq \dim T_{\iota(x)} \tilde{Z} = \dim_{\iota(x)} \tilde{Z} \leq \dim S_{\mathbb{C}} \leq \dim T_{\iota(x)} S_{\mathbb{C}}.$$

These inequalities must be equalities, so $\dim_{\iota(x)} \tilde{Z} = \dim S_{\mathbb{C}}$. But $S_{\mathbb{C}}$ is algebraically irreducible, hence analytically irreducible (4.13), and we get $\tilde{Z} = S_{\mathbb{C}}$ (4.12), a contradiction. □(Claim 3)

Let us now finish the proof of the proposition. We have to show that $X' \subseteq Z$. Consider the subset $f^{-1}(Z) \cap S_{\mathbb{C}} \subseteq \mathbb{C}^{2g}$. We have $\Phi(X) \subseteq Z$, and thus $\iota(X) \subseteq f^{-1}(Z)$ because of $\Phi = f \circ \iota$. Since $\iota(X) \subseteq S_{\mathbb{C}}$ by the definition of $S_{\mathbb{C}}$, we get $\iota(X) \subseteq f^{-1}(Z) \cap S_{\mathbb{C}}$. Moreover, $f^{-1}(Z) \cap S_{\mathbb{C}}$ is an analytic set in $\mathbb{C}^g$, so it follows from the third claim that $f^{-1}(Z) \cap S_{\mathbb{C}} = S_{\mathbb{C}}$, and therefore $f(S_{\mathbb{C}}) \subseteq Z$. Being a complex analytic subset of $\mathbb{C}^g$, the set Z is Euclidean closed in $\mathbb{C}^g$. Because $f(S_{\mathbb{C}})$ is Euclidean dense in X' by the second claim, we get $X' \subseteq Z$ as desired. □

4.2 Ax-Lindemann-Weierstraß

The main result of this subsection corresponds to the main theorem in the original work of Pila and Zannier (Theorem 2.1. in [PZ08]), yielding a proof of Manin-Mumford. We will present a proof of the following theorem:

Theorem 4.14 (Ax-Lindemann-Weierstraß)**.** *Let A be a complex Abelian variety of dimension g and let $\pi\colon \mathbb{C}^g \to A$ be its exponential map. Let V be a closed algebraic subvariety of A, and let Y be a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is a translate of an Abelian subvariety of A.*

Here, Y being a maximal irreducible complex algebraic subvariety in $\pi^{-1}(V)$ means that there is no irreducible complex algebraic subvariety Y' of $\mathbb{C}^g$ such that $Y \subsetneq Y' \subseteq Z$.

None of the mathematicians occuring in the name proved Ax-Lindemann-Weierstraß in this form. However, the theorem can be deduced from transcendence results due to Ax, and it can be viewed as a geometric analogue of the Lindemann-Weierstraß theorem in transcendental number theory. This and more is explained in more detail by Pila in his expository article *Functional transcendence via o-minimality* [Pil15]. Let us just very briefly elaborate on the background. There is a famous conjecture in transcendental number theory due to Schanuel:

Conjecture 4.15 (Schanuel)**.** *Suppose that $z_1, \dots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of $\mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$ over $\mathbb{Q}$ is at least n.*

Example 4.16. Suppose that the conjecture holds, then e and π are algebraically independent over $\mathbb{Q}$. Indeed, take $z_1 = 1$ and $z_2 = \pi i$ in the statement.

Schanuel's conjecture is unproved and has surprising connections to other fields of mathematics. For example, Macintyre and Wilkie proved that Schanuel's conjecture implies the decidability of $\mathbb{R}_{\exp}$, cf. [MW96]. A remarkable positive result is the power series version of Schanuel's conjecture due to Ax, which we have already encountered in the proof of Mann's Theorem (1.12): If $\gamma_1, \dots, \gamma_n \in \mathbb{C}[\![t]\!]$ are complex power series with no constant term that are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$, then the transcendence degree of $\mathbb{C}(\gamma_1, \dots, \gamma_n, \exp(\gamma_1), \dots, \exp(\gamma_n))$ over $\mathbb{C}$ is at least $n+1$, cf. Corolary 1 in [Ax71]. Shortly after his publication of the theorem, Ax related and expanded his results for applications to algebraic groups, cf. [Ax72]. The Ax-Lindemann-Weierstraß theorem as stated above (4.14) can be deduced from Theorem 3 in [Ax72]. However, this deduction is still nontrivial.

It is interesting to view Ax-Lindemann-Weierstraß in the context of transcendence results. However, we take a different path. We closely follow Orr and present a proof using the Pila-Wilkie Counting Theorem (3.13), cf. section 7 in [Orr15].

0. Setup, notations and outline. We are given a complex Abelian variety A of dimension g over $\mathbb{C}$, the exponential map $\pi\colon \mathbb{C}^g \to A$, a closed algebraic subvariety V of A, and a maximal irreducible algebraic subvariety contained in $\pi^{-1}(V)$, denoted by Y. The situation is displayed in the following diagram:

$$\begin{array}{ccccc} Y & \subseteq & \pi^{-1}(V) & \subseteq & \mathbb{C}^g \\ & & & & \downarrow{\scriptstyle \pi} \\ \pi(Y) & \subseteq & V & \subseteq & A \end{array}$$

Our goal is to show that $\pi(Y)$ is the translate of an Abelian subvariety of A. By carefully looking at the statement, we can already make two minor reductions.

(a) We can assume that $\dim Y > 0$. If $\dim Y = 0$, then Y is a point by irreducibility, and hence so also is $\pi(Y)$. Every point in A is the translate of the trivial Abelian subvariety.

(b) We can assume that V is the Zariski closure of $\pi(Y)$ in A. By definition, this closure is a complex algebraic subvariety of A. It is only left to show that Y is maximal among the irreducible algebraic subvarieties contained in $\pi^{-1}(\mathrm{cl}_A^{\mathbb{C}\text{-Zar}}\pi(Y))$. We have $Y \subseteq \pi^{-1}(\mathrm{cl}_A^{\mathbb{C}\text{-Zar}}\pi(Y))$ and $\mathrm{cl}_A^{\mathbb{C}\text{-Zar}}\pi(Y) \subseteq V$, hence

$$Y \subseteq \pi^{-1}(\mathrm{cl}_A^{\mathbb{C}\text{-Zar}}\pi(Y)) \subseteq \pi^{-1}(V).$$

Thus, if $Z \subseteq \mathbb{C}^g$ is an irreducible closed algebraic variety with $Y \subseteq Z \subseteq \pi^{-1}(\mathrm{cl}_A^{\mathbb{C}\text{-Zar}}\pi(Y))$, then also $Y \subseteq Z \subseteq \pi^{-1}(V)$ and it follows that $Y = Z$.

Let us fix some notations. As usual, we denote the kernel of π by Λ, which is a lattice in $\mathbb{C}^g$. Let $\lambda_1, \dots, \lambda_{2g}$ be a $\mathbb{Z}$-basis for Λ. We can define a height function on Λ by

$$\mathsf{H}\colon \Lambda \to \mathbb{N}, \quad \sum_{k=1}^{2g} a_k\lambda_k \mapsto \max\{|a_1|, \dots, |a_{2g}|\}.$$

The $\lambda_1, \dots, \lambda_{2g}$ form a basis for the real vector space $\mathbb{C}^g$, hence we can identify $\mathbb{R}^{2g}$ and $\mathbb{C}^g$ with the isomorphism of real vector spaces

$$\Psi\colon \mathbb{R}^{2g} \to \mathbb{C}^g, \quad (a_1, \dots, a_{2g}) \mapsto \sum_{k=1}^{2g} a_k\lambda_k$$

and call $X \subseteq \mathbb{C}^n$ definable if $\Psi^{-1}(X)$ is definable. Since automorphisms given by base change matrices $\mathbb{R}^{2g} \to \mathbb{R}^{2g}$ are definable in any expansion of $\mathbb{R}_{\mathrm{alg}}$, the choice of Ψ is irrelevant for questions of definability. However, the isomorphism Ψ is particularly suitable for us to work with, because under Ψ the lattice $\Lambda \subseteq \mathbb{C}^g$ corresponds to $\mathbb{Z}^{2g} \subseteq \mathbb{R}^{2g}$ and the height function above on Λ corresponds to the restriction of the classical height function on $\mathbb{Q}^{2g}$ to $\mathbb{Z}^{2g}$.

We also fix the interior of a fundamental domain $\mathcal{F}$ for Λ, say

$$\mathcal{F} := \left\{ \sum_{k=1}^{2g} a_k\lambda_k : a_k \in (0,1) \text{ for } 1 \le k \le 2g \right\}.$$

A central object of study in the proof will be the set

$$\Sigma := \{x \in \mathbb{C}^g : (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } Y + x \subseteq \pi^{-1}(V)\}.$$

The condition $(Y + x) \cap \mathcal{F}$ will ensure that Σ is definable, and the condition $Y + x \subseteq \pi^{-1}(V)$ will be exploited to see that the (setwise) stabiliser of Y in $\mathbb{C}^g$ has positive dimension. Relating this stabiliser to the stabiliser of V in A, we will be able to take quotients and finish the proof. In more detail, the strategy goes as follows.

1. Prove that the number of points on $\Sigma \cap \Lambda$ of height up to t grows at least linearly in t.

2. Show that Σ is definable in $\mathbb{R}_{\mathrm{an}}$. Conlcude with Pila-Wilkie and the lower bound from 1. that Σ contains a connected semialgebraic set of positive dimension with a lattice point.

3. Use this semialgebraic set to prove that the stabiliser Θ of Y in $\mathbb{C}$ has positive dimension. Show that the image of this stabiliser under π is the identity component B of the stabiliser of V in A.

4. Take the quotients of $\mathbb{C}^g$ by Θ and of A by B. Conclude the proof by applying the arguments from before to the quotients.

Let us begin with the proof.

1. A lower bound for lattice points in Σ. We show that the number of points on $\Sigma \cap \Lambda$ of height up to t grows at least linearly in t. If $x \in \Lambda$, then $Y + x \subseteq \pi^{-1}(V)$, because x is in the kernel of π and $Y \subseteq \pi^{-1}(V)$. Hence

$$\Sigma \cap \Lambda = \{x \in \Lambda : (Y + x) \cap \mathcal{F} \neq \emptyset\}.$$

Lemma 4.17. *There is $t_0 \in \mathbb{R}$ such that for all $t > t_0$ we have $\#\{x \in \Sigma \cap \Lambda : \mathsf{H}(x) \leq t\} \geq t/2$.*

Proof. Because it is an irreducible positive dimensional complex affine variety, Y is unbounded and path connected in the Euclidean topology (2.31,2.36). Therefore, we find a Euclidean continuous function $\gamma \colon [0, 1) \to Y$ with unbounded image. We consider

$$\Lambda_\gamma = \{x \in \Lambda : (\mathcal{F} - x) \cap \operatorname{im} \gamma \neq \emptyset\},$$

We have $\Lambda_\gamma \subseteq \Sigma \cap \Lambda$, because $(\mathcal{F} - x) \cap \operatorname{im} \gamma \neq \emptyset$ implies that $(Y + x) \cap \mathcal{F} \neq \emptyset$. If the image of γ crosses from some $\mathcal{F} - x$ to an adjacent $\mathcal{F} - x'$, then the heights $\mathsf{H}(x)$ and $\mathsf{H}(x')$ differ by at most 1. Thus

$$\{\mathsf{H}(x) : x \in \Lambda_\gamma\}$$

is a set of consecutive natural numbers. Because $\operatorname{im} \gamma$ is unbounded in $\mathbb{C}^g$, the set Λ_γ contains points of arbitratily large height. Take $x_0 \in \Lambda_\gamma$, then for all $h \geq \mathsf{H}(x_0)$, the set Λ_γ contains at least one point of height h. Set $t_0 := 2\mathsf{H}(x_0)$, then for all $t > t_0$ we have

$$\#\{x \in \Lambda_\gamma : \mathsf{H}(x) \leq t\} = \sum_{k=0}^{t} \#\{x \in \Lambda_\gamma : \mathsf{H}(x) = k\} \geq \sum_{k=\mathsf{H}(x_0)+1}^{t} 1 = t - \mathsf{H}(x_0) > t/2.$$

Since $\Lambda_\gamma \subseteq \Sigma \cap \Lambda$, this proves the lemma. □

Remark. There is a gap in this proof. We have chosen $\mathcal{F}$ to be open, so a priori, γ could also run inside the boundaries of the open boxes, and must not cross *from one $\mathcal{F} - x$ to an adjacent $\mathcal{F} - x'$*. We cannot easily take the $\mathcal{F}$ to be half open (take $[0, 1)$ in the definition and not $(0, 1)$), because then an *identity theorem* argument in the next step would fail. The author of this thesis and Dr. Orr have discussed the problem did not (yet) find a solution. The gap might be fixable by a more careful choice of γ. In fact, any two points on Y can be joined by an irreducible curve inside Y (2.36). Note however, that such curves are no longer paths in the usual topological sense, because they are 1-dimensional as *complex* varieties. However, it would still be possible to take an unbounded sequence $(y_i)_{i \in \mathbb{N}}$ of points in Y, join y_i and y_{i+1} with an irreducible curve C_i in Y, and replace Λ_γ with the points $x \in \Lambda$ such that $\mathcal{F} - x$ intersects some C_i nontrivially.

2. A semialgebraic set of positive dimension in Σ. We prove that Σ contains a connected semialgebraic set of positive dimension that contains a lattice point. So we will prove that $\Sigma^{\text{alg}} \cap \Lambda$, or more precisely $\Psi^{-1}(\Sigma)^{\text{alg}} \cap \mathbb{Z}^{2g}$, is nonempty. We already know that there are *many* points in $\Sigma \cap \Lambda$ in the sense of the lower bound from the first step. The idea is now to prove with Pila-Wilkie that there are *few* points in $\Psi^{-1}(\Sigma)^{\text{trans}} \cap \mathbb{Z}^{2g}$. Let us first prove that Σ is definable.

Lemma 4.18. *The subset Σ of $\mathbb{C}^g$ is definable in $\mathbb{R}_{\text{an}}$.*

Proof. We prove that $\Psi^{-1}(\Sigma) \subseteq \mathbb{R}^{2g}$ is definable in $\mathbb{R}_{\text{an}}$. Recall the definition

$$\Sigma = \{x \in \mathbb{C}^g : (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } Y + x \subseteq \pi^{-1}(V)\}.$$

Because in the second condition we have the *unrestricted* analytic function $\pi\colon \mathbb{C}^g \to A$, the definabilitry of Σ in $\mathbb{R}_{\text{an}}$ is nontrivial. The trick is to show that Σ equals

$$\Sigma' := \{x \in \mathbb{C}^g : (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } (Y + x) \cap \mathcal{F} \subseteq \pi^{-1}(V) \cap \mathcal{F}\},$$

which we will see to be definable because $(Y+x) \cap \mathcal{F}$ and $\pi|_{\mathcal{F}}^{-1}(V)$ are sets defined by analytic functions *restricted* to $\mathcal{F}$. Let us first show that indeed $\Sigma = \Sigma'$ holds. The inclusion $\Sigma \subseteq \Sigma'$ is immeadiate. For $\Sigma' \subseteq \Sigma$, take $x \in \mathbb{C}^g$ such that $(Y+x)\cap\mathcal{F} \neq \emptyset$ and $(Y+x)\cap\mathcal{F} \subseteq \pi^{-1}(V)\cap\mathcal{F}$. We have to show that $Y + x \subseteq \pi^{-1}(V)$. Let $g_1, \dots, g_s \in \mathbb{C}[X_0, \dots, X_n]$ be homogeneous polynomials defining V for some chosen projective embedding. From $(Y + x) \cap \mathcal{F} \subseteq \pi^{-1}(V)$, it follows that the analytic functions $g_j \circ \pi$ vanish on $(Y + x) \cap \mathcal{F}$ for all $j \in \{1, \dots, s\}$. But $Y + x$ is irreducible as an algebraic set, hence also irreducible as an analytic set (4.13). This is equivalent to the fact that the regular points of $Y + x$ are Euclidean connected (cf. the proof of (4.13)). The functions $g_j \circ \pi$ restrict to analytic functions of connected complex manifolds $(Y + x)^{\text{reg}} \to \mathbb{C}$, which vanish on the nonempty open subset $(Y + x)^{\text{reg}} \cap \mathcal{F}$, thus already on all of $(Y + x)^{\text{reg}}$ by the identity theorem for complex manifolds (Theorem 1.8. in [Ada13]). Since $(Y + x)^{\text{reg}}$ is Euclidean open and dense in $Y + x$ (4.9), it follows that the $g_j \circ \pi$ vanish on all of $Y + x$. This proves $\pi(Y + x) \subseteq V$, and thus $\Sigma = \Sigma'$.

Let us now verify that $\Psi^{-1}(\Sigma')$ is definable in $\mathbb{R}_{\text{an}}$. It is the intersection of the preimages of

$$\Sigma_1 := \{x \in \mathbb{C}^g : (Y + x) \cap \mathcal{F} \neq \emptyset\} \quad \text{and} \quad \Sigma_2 = \{x \in \mathbb{C}^g : (Y + x) \cap \mathcal{F} \subseteq \pi^{-1}(V) \cap \mathcal{F}\}.$$

In the following, we shorten notation by using logical symbols before knowing that we actually work with $\mathcal{L}_{\text{an}}$-formulas. The set Σ_1 is even definable in $\mathbb{R}_{\text{alg}}$. Indeed, we have

$$\begin{aligned}\Psi^{-1}(\Sigma_1) &= \{a \in \mathbb{R}^{2g} : (\Psi^{-1}(Y) + a) \cap \Psi^{-1}(\mathcal{F}) \neq \emptyset\} \\ &= \{a \in \mathbb{R}^{2g} : \mathbb{R}_{\text{alg}} \models \exists b (b \in \Psi^{-1}(Y) \wedge b + a \in (0,1)^{2g})\},\end{aligned}$$

and since $(0,1)^{2g}$ is definable in $\mathbb{R}_{\text{alg}}$, it suffices to show that the same holds for $\Psi^{-1}(Y)$. Let $f_1, \dots, f_r$ in $\mathbb{C}[X_1, \dots, X_n]$ be polynomials whose zero set is Y, then we have

$$\Psi^{-1}(Y) = \left\{ b \in \mathbb{R}^{2g} : \mathbb{R}_{\text{alg}} \models \bigwedge\nolimits_{j=1}^{r} f_j(\Psi(b)) = 0 \right\}.$$

Identifying $\mathbb{C}$ and $\mathbb{R}^2$ with real and imaginary parts, the maps $f_j \circ \Psi\colon \mathbb{R}^{2g} \to \mathbb{R}^2$ are polynomials in each coordinate, hence their zero sets are definable in $\mathbb{R}_{\text{alg}}$. So $\Psi^{-1}(\Sigma_1)$ is definable in $\mathbb{R}_{\text{alg}}$.

Similar arguments work for Σ_2. Unfolding the definition, we see that $\Psi^{-1}(\Sigma_2)$ equals

$$\{a \in \mathbb{R}^{2g} : \mathbb{R}_{\text{an}} \models \forall c (\exists b (b \in \Psi^{-1}(Y) \wedge c = b + a \wedge c \in \Psi^{-1}(\mathcal{F})) \to c \in \Psi^{-1}(\pi^{-1}(V) \cap \mathcal{F}))\}.$$

As before, $\Psi^{-1}(Y)$ and $\Psi^{-1}(\mathcal{F})$ are definable in $\mathbb{R}_{\text{alg}}$, and it remains to show that

$$\Psi^{-1}(\pi^{-1}(V) \cap \mathcal{F}) = \{a \in \mathbb{R}^{2g} : \pi|_{\mathcal{F}}(\Psi(x)) \in V\}$$

is definable in $\mathbb{R}_{\text{an}}$. Take homogeneous polynomials $g_1, \dots, g_s \in \mathbb{C}[X_0, \dots, X_n]$ defining V for some projective embedding. Separating real and imaginary parts, the functions

$$g_j \circ \pi \circ \Psi\colon \mathbb{R}^{2g} \to \mathbb{C} \cong \mathbb{R}^2$$

are real analytic on an open neighbourhood of $[0,1]^{2g}$ in every coordinate. By compactness, we find finitely many open boxes covering $[0,1]^{2g}$ on each of which these functions are given by convergent power series in every coordinate. Restricting further to $(0,1)^{2g} = \Psi^{-1}(\mathcal{F})$, we obtain definablility of the functions and hence their zero sets in $\mathbb{R}_{\text{an}}$. □

Corollary 4.19. *The set $\Sigma^{\text{alg}} \cap \Lambda$ is nonempty, i.e. $\Psi^{-1}(\Sigma)^{\text{alg}} \cap \mathbb{Z}^{2g}$ is nonempty.*

Proof. By the previous lemma (4.18), the set $\pi^{-1}(\Sigma)$ is definable in $\mathbb{R}_{\mathrm{an}}$ and we can apply Pila-Wilkie with an ε strictly smaller than 1, say $\varepsilon = 1/2$. So we find a constant C such that for all $t \geq 1$, we have

$$\#\{a \in \Psi^{-1}(\Sigma)^{\mathrm{trans}} \cap \mathbb{Q}^{2g} : \mathsf{H}(a) \leq t\} \leq Ct^{1/2}.$$

and in particular

$$\#\{a \in \Psi^{-1}(\Sigma)^{\mathrm{trans}} \cap \mathbb{Z}^{2g} : \mathsf{H}(a) \leq t\} \leq Ct^{1/2}. \tag{1}$$

On the other hand, we know that for sufficiently large t, we have

$$\#\{a \in \Psi^{-1}(\Sigma) \cap \mathbb{Z}^{2g} : \mathsf{H}(a) \leq t\} = \#\{x \in \Sigma \cap \Lambda : \mathsf{H}(x) \leq t\} \geq t/2 \tag{2}$$

by the first step (4.17). Comparing the bounds (1) and (2), we see that

$$\Psi^{-1}(\Sigma)^{\mathrm{alg}} \cap \mathbb{Z}^{2g} = (\Psi^{-1}(\Sigma) \setminus \Psi^{-1}(\Sigma)^{\mathrm{trans}}) \cap \mathbb{Z}^{2g}$$

is nonempty. □

Corollary 4.20. *There is a positive dimensional connected semialgebraic subset $W \subseteq \Sigma$ with a lattice point $w_0 \in W \cap \Lambda$. We can also assume that W is irreducible.*

Proof. The first part is the content of the previous corollary (4.19). We can assume that W is irreducible because $\mathbb{R}^{2n}$ and thus also W is Noetherian with the Zariski topology. □

3. The stabilisers of Y in $\mathbb{C}^g$ and of V in A. Let Θ be the stabiliser of Y in $\mathbb{C}^g$, i.e.

$$\Theta = \{x \in \mathbb{C}^g : x + y \in Y \text{ for all } y \in Y\},$$

which is an additive subgroup of $\mathbb{C}^g$. We even have more:

Lemma 4.21. *The additive subgroup Θ of $\mathbb{C}^g$ is an algebraic set and in fact a complex vector subspace of $\mathbb{C}^g$.*

Proof. We have $\Theta = \{x \in \mathbb{C}^g : x+y \in Y \text{ for all } y \in Y\} = \bigcap_{y \in Y} Y - y$, which is an intersection of Zariski closed sets, hence Zariski closed.

To prove that Θ is a subspace, let $x \in \Theta \setminus \{0\}$. We have to show that $zx \in \Theta$ for all $z \in \mathbb{C}$. Let L be the 1-dimensional vector subspace of $\mathbb{C}^g$ spanned by x, which is also a 1-dimensional irreducible algebraic set. Since $x \neq 0$, the set $\{nx : n \in \mathbb{Z}\}$ is an infinite subset of the algebraic set $L \cap \Theta$. But this set has Krull dimension at most 1, so $L \cap \Theta = L$, and thus $L \subseteq \Theta$. □

We can also consider the stabiliser of V in A. In general, this will only be an algebraic subgroup and not be an Abelian subvariety. But the Zariski identity component of this subgroup is an Abelian subvariety (2.29). Let us denote the identity component of the stabiliser of V in A by B. We will prove that Θ and B are positive dimensional and that $\pi(\Theta) = B$. Let us begin with proving that Θ has positive dimension. As it is a complex vector subspace of $\mathbb{C}^g$, it suffices to show that it is not discrete. The idea is to show that $W - w_0 \subseteq \Theta$.

Lemma 4.22. *We have $W - w_0 \subseteq \Theta$, In fact, if $W \subseteq \Sigma$ is any connected irreducible semialgebraic set with some point $w_0 \in W \cap \Lambda$, then $Y + W - w_0 = Y$.*

Proof. For all $w \in W$ we have $w \in \Sigma$, hence $Y + w \subseteq \pi^{-1}(V)$ by the definition of Σ. We can conclude that $Y + W \subseteq \pi^{-1}(V)$. Since w_0 is in Λ, which is the kernel of π, we get

$$Y + W - w_0 \subseteq \pi^{-1}(V).$$

Note that $Y + W - w_0$ is a connected irreducible real semialgebraic set. Certainly, $Y + W - w_0$ is semialgebraic. As an irreducible complex algebraic set, Y is Euclidean connected in $\mathbb{C}^g$

(2.35). Since also W is Euclidean connected in $\mathbb{C}^g$, we get that $Y \times W \subseteq \mathbb{C}^g \times \mathbb{C}^g$ is Euclidean connected. Because $+ \colon \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}^g$ is Euclidean continuous, the image $Y + W$ is Euclidean connected. Irreducibility follows by the same pattern, using that $+ \colon \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ is Zariski continuous and that products and continuous images of irreducible spaces are irreducible.

Since $\pi^{-1}(V)$ is a complex analytic set, we can now apply the main result of the previous subsection (4.2) to see that there is an irreducible complex algebraic set $Y' \subseteq \mathbb{C}^g$ with

$$Y + W - w_0 \subseteq Y' \subseteq \pi^{-1}(V).$$

However, Y is assumed to be maximal among the closed irreducible algebraic varieties contained in $\pi^{-1}(V)$ and we have $Y \subseteq Y + W - w_0$, so it follows that $Y = Y + W - w_0 = Y'$. The first of these equalities is the claim of the lemma. □

We conclude that Θ is a positive dimensional subspace of $\mathbb{C}^g$, because it is not discrete by the previous lemma (4.22).

Lemma 4.23. *We have $\pi(\Theta) = B$.*

Proof. Let us first show that $\pi(\Theta) \subseteq B$. Let $x \in \Theta$, i.e. $Y + x = Y$. Since $Y \subseteq \pi^{-1}(V)$, we get $Y \subseteq \pi^{-1}(V) - x$, and thus $\pi(Y) \subseteq V - \pi(x)$. Since we also have $\pi(Y) \subseteq V$, we obtain

$$\pi(Y) \subseteq V \cap (V - \pi(x)).$$

As V is a complex algebraic subvariety of A, so also is $V \cap (V - \pi(x))$. But we assumed that V is the Zariski closure of $\pi(Y)$ in A, so from the above inclusion it already follows that

$$V \subseteq V \cap (V - \pi(x)),$$

and hence $V \subseteq V - \pi(x)$. Applying the same argument to $-x$, we see that $V \subseteq V + \pi(x)$. This yields the inclusions

$$V - \pi(x) \subseteq V \subseteq V - \pi(x),$$

which have to be equalities. We conclude that $V + \pi(x) = V$, so $\pi(\Theta)$ is a subset of the stabiliser of V in A. For the first inclusion, it remains to show that $\pi(\Theta)$ is contained in the identity component the stabiliser of V in A. Being a subspace of $\mathbb{C}^g$, the stabiliser Θ is Euclidean connected, hence also its image $\pi(\Theta)$ is Euclidean connected. Because the Euclidean topology is finer than the Zariski topology, we see that $\pi(\Theta)$ is also Zariski connected. Moreover, π is a group homomorphism and $0 \in \Theta$, thus the identity element of A is in $\pi(\Theta)$. This completes the proof of $\pi(\Theta) \subseteq B$.

Let us prove the other inclusion $B \subseteq \pi(\Theta)$. We know that $\pi^{-1}(B)$ is a Lie subgroup of the additive group $\mathbb{C}^g$ and may take its Euclidean identity component Θ', which is again an Lie subgroup of $\mathbb{C}^g$ (it is additively closed because $+$ is Euclidean continuous). As for Θ, in fact more holds. By considering their exponential map, it can be proved that every connected Abelian complex Lie group is a quotient of some $\mathbb{C}^m$ by a lattice. Since $\Theta' \subseteq \mathbb{C}^g$ must be torsion free in the present case, the lattice must be trivial, and Θ' is a complex subspace.

The image $\pi(\Theta')$ is an Lie subgroup of B. Because π is a covering map, taking images and preimages under π does not change the dimension of manifolds. Also taking the Euclidean identity component of a Lie group does not change the dimension. Therefore, $\pi(\Theta')$ is a Lie subgroup of B of the same dimension as B, and both of these are finite dimensional and connected. This in fact suffices for $B = \pi(\Theta')$, as we will briefly argue. From the fact that $\pi(\Theta')$ and B have the same dimension, it follows that $\pi'(\Theta')$ is Euclidean open in B, but then $\pi'(\Theta')$ already equals B, because $B \setminus \pi(\Theta')$ is a union of translates of B, hence open too. Since B is connected, we conclude that $B = \pi(\Theta')$.

To prove that $B \subseteq \pi(\Theta)$ it thus suffices to show that $\Theta' \subseteq \Theta$. We know that B stabilises V, hence $\pi^{-1}(B)$ stabilises $\pi^{-1}(V)$ and in particular $\Theta' \subseteq \pi^{-1}(B)$ stabilises $\pi^{-1}(V)$. Moreover, we have $Y \subseteq \pi^{-1}(V)$, so

$$Y + \Theta' \subseteq \pi^{-1}(V).$$

As a complex vector space, Θ' is an irreducible algebraic set. Thus also, $Y + \Theta'$ is an irreducible complex algebraic set containing Y, and the maximality of Y implies that $Y + \Theta' = Y$. This proves $\Theta' \subseteq \Theta$. In conclusion, we have $\pi(\Theta) = B$ as claimed. □

Corollary 4.24. *The Abelian subvariety B of V has positive dimension.*

Proof. We know that Θ is a positive dimensional vector subspace of $\mathbb{C}^g$ and that $\pi\colon \mathbb{C}^g \to A$ is a covering map with $\pi(\Theta) = B$. So B is positive dimensional as a complex manifold, and thus also positive dimensional as an Abelian variety. □

4. End of the proof. We consider the quotient Abelian variety A/B, as well as the quotient vector space $\mathbb{C}^g/\Theta$ (2.56). Let $q\colon A \to A/B$ and $\tilde{q}\colon \mathbb{C}^g \to \mathbb{C}^g/\Theta$ denote the respective quotient maps. Since $\pi(\Theta) = B$ by the previous step, we have $\Theta \subseteq \ker(q \circ \pi)$, and get an induced map $\pi'\colon \mathbb{C}^g/\Theta \to A/B$ such that the diagram

$$\begin{tikzcd}
\mathbb{C}^g \arrow[r, "\tilde{q}"] \arrow[d, "\pi"'] & \mathbb{C}^g/\Theta \arrow[d, "\pi'"] \\
A \arrow[r, "q"'] & A/B
\end{tikzcd}$$

commutes. Moreover, π' is the exponential map for $\mathbb{C}^g/\Theta$ (2.56). Our goal was to prove that $\pi(Y)$ is the translate of an Abelian subvariety of A. We prove that $\pi(Y)$ is a translate of B.

Let $V' := q(V) \subseteq A/B$ and let Y' be the Zariski closure of $\tilde{q}(Y)$ in $\mathbb{C}^g/\Theta$. Since $\tilde{q}$ is linear, hence regular, Y' is irreducible and closed. Formally, Y' is an abstract algebraic variety, but we can identify $\mathbb{C}^g/\Theta$ with $\mathbb{C}^{g-\dim\Theta}$ to treat it as an algebraic set.

Lemma 4.25. *We have $V = q^{-1}(V')$.*

Proof. We only have to prove that $q^{-1}(q(V)) \subseteq V$. Let $x \in q^{-1}(q(V)) \subseteq V$, so

$$x + B = q(x) \in q(V) = \{v + B : v \in V\}$$

and we find $v \in V$ with $x + B = v + B$. Because we chose B to be the identity component, in particular we have $x \in v + B$. But since B is a subset of the stabiliser of V, we have $v + B \subseteq V$, hence $x \in V$. □

Lemma 4.26.

(a) The subvariety V' of A/B is the Zariski closure of $\pi'(Y')$ in A/B.

(b) The subvariety Y' of $\mathbb{C}^g/\Theta$ is a maximal irreducible algebraic subvariety contained in $\pi'^{-1}(V')$.

Proof. To prove (a), let Z' be Zariski closed in A/B with $\pi'(Y') \subseteq Z' \subseteq V'$. Then

$$q^{-1}(\pi'(Y')) \subseteq q^{-1}(Z') \subseteq q^{-1}(V')$$

However, $q^{-1}(V') = V$ by the previous lemma (4.25) and by chasing the above diagram, we see that for all $S \subseteq \mathbb{C}^g/\Theta$ we have $\pi(\tilde{q}^{-1}(S)) \subseteq q^{-1}(\pi'(S))$. Applying this to $S = Y'$ and using that $\pi(Y) \subseteq \pi(\tilde{q}^{-1}(Y'))$, we obtain

$$\pi(Y) \subseteq q^{-1}(Z') \subseteq V.$$

However, the quotient map q of Abelian varieties is regular, hence $q^{-1}(Z')$ is a subvariety of A and the fact that V is the closure of $\pi(Y)$ in A implies that $q^{-1}(Z') = V$, yielding $Z' = q(V) = V'$. Hence (a) holds.

For (b), let Z' be an irreducible algebraic subvariety of $\mathbb{C}^g/\Theta$ with $Y' \subseteq Z' \subseteq \pi'^{-1}(V')$. We then have $Y \subseteq \tilde{q}^{-1}(Y') \subseteq \tilde{q}^{-1}(Z')$ and by $q^{-1}(V') = V$ and the commutativity of the above diagram $\tilde{q}^{-1}(Z') \subseteq \pi^{-1}(V)$, thus

$$Y \subseteq \tilde{q}^{-1}(Z') \subseteq \pi^{-1}(V).$$

Because $\tilde{q}$ is linear, $\tilde{q}^{-1}(Z')$ is an algebraic set in $\mathbb{C}^g$. By its maximality, Y must be one of the irreducible components of the algebraic set $\tilde{q}^{-1}(Z')$. Hence, $Y' = \mathrm{cl}^{\mathbb{C}\text{-Zar}}(\tilde{q}(Y))$ must be among the irreducible components of $\tilde{q}(\tilde{q}^{-1}(Z')) = Z'$. However, Z' is irreducible by assumption, so $Y' = Z'$ as desired. □

We prove a final corollary before concluding Ax-Lindemann-Weierstraß.

Corollary 4.27. *The algebraic set Y' is a point.*

Proof. Since Y' is irreducible, it suffices to prove that $\dim Y' = 0$. Assume towards a contradiction that $\dim Y' > 0$. Since V' is the Zariski closure of $\pi'(Y')$ and Y' is a maximal irreducible algebraic subvariety contained in $\pi'^{-1}(V')$ by the previous lemma (4.26), we can apply the steps 1 to 3 to $(A/B, V', Y')$. This yields the conclusion that the identity component B' of the stabiliser of V' in A' has positive dimension. But $q^{-1}(B')$ stabilises V, so this contradicts the fact that $\ker q = B$ is the identity component of the stabiliser of V. □

We now know that $Y' = \tilde{q}(Y)$ is a point, hence $\pi'(\tilde{q}(Y)) = q(\pi(Y))$ is a point, say $a+B \in A/B$ for some $a \in \pi(Y)$. So for all $x \in \pi(Y)$, we have $x+B = a+B$, and in particular $x+0 \in a+B$. Thus, $\pi(Y) \subseteq a + B$ follows. For the other inclusion, let $y \in Y$ such that $a = \pi(y)$ and take any $b = \pi(\vartheta) \in \pi(\Theta) = B$ with $\vartheta \in \Theta$, then $a + b = \pi(y + \vartheta) \in \pi(Y)$. Hence, we also have $a + B \subseteq \pi(Y)$ and conclude that $a + B = \pi(Y)$.

As desired, $\pi(Y)$ is a translate of an Abelian subvariety of A. □(Theorem 4.14)

4.3 Concluding Manin-Mumford

We will now deduce Manin-Mumford. We have discussed two main ingredients in detail:

Theorem (Pila-Wilkie (3.13))**.** *Let X be definable in an o-minimal expansion of $\mathbb{R}_{\mathrm{alg}}$. Then for all $\varepsilon > 0$ there exists a constant C, only depending on X and ε, such that $\mathsf{N}(X^{\mathrm{trans}}, t) \leq Ct^{\varepsilon}$ for all $t \in \mathbb{R}_{\geq 0}$.*

Theorem (Ax-Lindemann-Weierstraß (4.14))**.** *Let A be a complex Abelian variety of dimension g and let $\pi\colon \mathbb{C}^g \to A$ be its exponential map. Let V be a closed algebraic subvariety of A, and let Y be a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is a translate of an Abelian subvariety of A.*

We will also once more use the main result from the beginning of the present section.

Proposition (4.2)**.** *Let Z be a complex analytic subset of $\mathbb{C}^g$ and let $W \subseteq \mathbb{C}^g$ be a connected irreducible semialgebraic subset such that $W \subseteq Z$. Then there exists an irreducible complex algebraic set X' in $\mathbb{C}^g$ such that $W \subseteq X' \subseteq Z$.*

Below, we will refer to this proposition as the *Comparison Result.*

To prove Manin-Mumford, there is only one ingredient missing. Recall how we played the upper bound of Pila-Wilkie against a lower bound from Galois theory in the first section on

Mann's Theorem (1.10). To show that $Y \cap \mathbb{G}^n_{\mathrm{tor}}$ is finite, we first reduced to the case that Y is defined over a number field k. Then for a tuple of roots of unity $P = (e^{2\pi i a_1/b_1}, \ldots, e^{2\pi i a_n/b_n})$ on Y, we considered the a field extension $k(P) := k(e^{2\pi i a_1/b_1}, \ldots, e^{2\pi i a_n/b_n})$ and proved that the respective Galois conjugates of P are also torsion points of the same order as P.

Similar arguments work in the present case of Abelian varieties, where we want to understand $V \cap A_{\mathrm{tor}}$. However, it is substantially more difficult to obtain the lower bound. Let us just mention the central ideas. As before in this section, we can fix some projective embedding of A into $\mathbb{P}^n(\mathbb{C})$. Then A is the zero set of homogeneous polynomials with coefficients in $\mathbb{C}$. Using nontrivial *specialisation* arguments, it can be proved that we may assume that A is defined over a number field. Of course, we could also add this assumption to the statement of the theorem. Given that A is defined over k, it can be proved that the coordinates of torsion points of A are algebraic over k. Similar to the argument we gave in the multiplicative case, we can then in fact also suppose that the closed subvariety V of A is defined over a number field. Taking the composite of these fields, we may suppose that A and V are defined over the same number field k. For a torsion point $P = [P_1 : \ldots : P_n]$ of A on V, we can consider the finite field extension $k(P) = k(P_1, \ldots, P_n)$ of k and for every $\sigma \in \mathrm{Aut}(k(P)/k)$, also $\sigma(P)$ is a torsion point on V. Moreover, it can be proved that the Galois conjugates of P have the same order as P. The relevant bound is then provided by Masser's Theorem:

Theorem 4.28 (Masser [Mas84])**.** *Let A be an Abelian variety of dimension g defined over a number field k. Then there exists a constant c depending on A and k, and $\rho > 0$ depending only on the dimension g, such that for all torsion points P of A of order t, we have*

$$[k(P) : k] \geq c t^{\rho}.$$

Alongside Masser's original paper [Mas84], we refer to an expository paper by Habegger [Hab15]. In Habegger's article, the bound is proved for elliptic curves, and some of the reductions outlined above are explained in more detail.

With these ingredients at hand, we can finally prove Manin-Mumford.

Theorem 4.29. *Let A be a complex Abelian variety and let $V \subseteq A$ be a closed algebraic subvariety. Suppose that V does not contain any translate of a positive dimensional Abelian subvariety of A. Then V contains only finitely many torsion points of A.*

Proof. Let $\pi\colon \mathbb{C}^g \to A$ be the exponential map and let Λ be the kernel of π, which is a lattice in $\mathbb{C}^g$. Choose a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_{2g}$ and the respective fundamental domain $\mathcal{F}$ for Λ. As in the previous subsection, we identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$ via the basis $\lambda_1, \ldots, \lambda_{2g}$, i.e. using the isomorphism of real vector spaces

$$\Psi\colon \mathbb{R}^{2g} \to \mathbb{C}^g, \quad (a_1, \ldots, a_{2g}) \mapsto \sum_{k=1}^{2g} a_k \lambda_k.$$

Note that we chose $\mathcal{F}$ so that $\Psi^{-1}(\mathcal{F}) = [0,1)^{2g}$, not $\Psi^{-1}(\mathcal{F}) = (0,1)^{2g}$ as in the previous part. A fundamental domain of a lattice is in particular a fundamental domain for its action, hence the exponential map π restrcits to a bijection between $\mathcal{F}$ and $A \cong \mathbb{C}^g/\Lambda$. We define

$$X := \pi|_{\mathcal{F}}^{-1}(V).$$

By the description of the torsion points on a complex Abelian variety (2.50), the exponential map π restricts to a bijection between the rational points on $\Psi^{-1}(X)$ and the torsion points of A on V. Abusing notation, we also write X for $\Psi^{-1}(X)$. Similar to the multiplicative case (1.6), we can relate heights of elements in $X \cap \mathbb{Q}^{2g} = X(\mathbb{Q})$ to the orders of their images in $V \cap A_{\mathrm{tor}}$. For $q \in X(\mathbb{Q})$, say $q = (a_1/b_1, \ldots, a_{2g}/b_{2g})$ with $a_j \in \mathbb{N}$, $b_j \in \mathbb{N} \setminus \{0\}$ coprime and $b_j = 1$ for $a_j = 0$, we have $\mathsf{H}(q) = \max\{b_1, \dots, b_{2g}\}$, while the order of $\Psi(q)$ in A is the same as the order of q in $(\mathbb{Q}/\mathbb{Z})^{2g}$ (2.50), i.e. the least common multiple of the $b_1, \dots, b_{2g}$. In particular, as in the proof of Mann's theorem,

$$\mathsf{H}(q) \leq \operatorname{ord}(\pi(q)) \leq \mathsf{H}(q)^{2g},$$

and to prove that $V \cap A_{\mathrm{tor}}$ is finite it suffices to show that there is an upper bound for the heights of rational points on X. In the previous subsection, we proved with a straightforward compactness argument that $\pi^{-1}(V) \cap (0,1)^g$ is definable in $\mathbb{R}_{\mathrm{an}}$ (at the end of the proof of (4.18)). The same argument shows that

$$\pi^{-1}(V) \cap [0,1)^{2g} = X$$

is definable in $\mathbb{R}_{\mathrm{an}}$, so we conclude that X is definable in $\mathbb{R}_{\mathrm{an}}$.

Analogously to our proof of Mann's Theorem (1.13), we define the special locus of V in A by

$$\operatorname{SpL}(V) := \bigcup_{(a,H) \in \mathcal{S}} a + H,$$

where $\mathcal{S}$ is the set of pairs (a, H) such that $a \in A$ and H is a closed positive dimensional Abelian subvariety of A with $a + H \subseteq V$.

Claim. We have $\pi(X^{\mathrm{alg}}) \subseteq \operatorname{SpL}(V)$.

Proof of the Claim. Let $x \in X^{\mathrm{alg}}$. By the definition of X^{alg}, there is a positive dimensional connected semialgebraic subset $W \subseteq X$ with $x \in W$. Because W is Noetherian with the real Zariski topology from $\mathbb{R}^{2g}$, we can assume that W is irreducible, cf. (4.20). As $X \subseteq \pi^{-1}(V)$, we also have $W \subseteq \pi^{-1}(V)$, and thus W is a connected irreducible semialgebraic set contained in the analytic set $\pi^{-1}(V)$. By the Comparison Result (4.2), we find an irreducible complex algebraic set X' in $\mathbb{C}^g$ with

$$W \subseteq X' \subseteq \pi^{-1}(V).$$

Since $\mathbb{C}^g$ has finite Krull dimension, we can also assume that X' is a maximal irreducible set contained in $\pi^{-1}(V)$. From Ax-Lindemann-Weierstraß (4.14), we obtain that $\pi(X')$ is a translate of an Abelian subvariety of A. But W is infinite, hence X' is infinite, and thus also $\pi(X')$ is infinite. We conclude that $\pi(x)$ is contained in a translate of a positive dimensional Abelian subvariety of A contained in V, that is, $\pi(x) \in \operatorname{SpL}(V)$. □(Claim)

Under the present assumptions of Manin-Mumford, the set V does not contain any translate of a positive dimensional Abelian subvariety, hence $\operatorname{SpL}(V) = \emptyset$, and we conclude that X^{alg} is empty. For the statement of Manin-Mumford, it now suffices consider X^{trans}.

We can assume that A and V are defined over a number field k and apply Masser's Theorem (4.28). There exist constants $c > 0$ and $\rho > 0$ such that for every torsion point P on A of order t we have

$$[k(P) : k] \geq c t^{\rho}.$$

The Galois conjugates of torsion points in V of order t are also torsion points in V of order t. Thus, if there exists one torsion point of order t on $V \cap A_{\mathrm{tor}}$, then there exist at least ct^{ρ} many such points.

On the other hand, taking $\varepsilon := \rho/2$ in the Pila-Wilkie Theorem (3.13), we get a constant C such that

$$\mathsf{N}(X^{\mathrm{trans}}, t) \leq C t^{\rho/2}$$

for all $t \in \mathbb{R}_{\geq 0}$. Our objective is to show that the order of torsion points of A on V is bounded. Let $t \in \mathbb{N}_{\geq 1}$ be the order of a torsion point $P \in V \cap A_{\mathrm{tor}}$. Then

$$\mathsf{N}(X^{\mathrm{trans}}, t) \overset{(1)}{=} \#\{q \in X^{\mathrm{trans}}(\mathbb{Q}) : H(q) \leq t\} \overset{(2)}{\geq} \#\{Q \in V \cap A_{\mathrm{tor}} : \operatorname{ord}(Q) = t\} \overset{(3)}{\geq} ct^{\rho}.$$

The equality (1) holds by definition. The inequality (2) holds since π restricts to a bijection $X^{\mathrm{trans}}(\mathbb{Q}) \to V \cap A_{\mathrm{tor}}$, and because $\mathsf{H}(q) \leq \mathrm{ord}(\pi(q))$ for all $q \in X^{\mathrm{trans}}(\mathbb{Q})$. The inequality (3) holds by Masser's Theorem, since we assumed that there exists a torsion point P on V of order n. On the other hand, by Pila-Wilkie, we have $\mathsf{N}(X^{\mathrm{trans}}, n) \leq Ct^{\rho/2}$. We obtain

$$ct^{\rho} \leq Ct^{\rho/2},$$

which holds if and only if $t \leq (C/c)^{2/\rho}$. We found an upper bound for the orders of torsion points of A on V, and conclude that V contains only finitely many torsion points of A. $\square$

References

[Ada13] J. Adamus. Complex analytic geometry, lecture notes part 1. Available at: https://www.math.uwo.ca/faculty/adamus/adamus_publications/AGI.pdf, 2013.

[Ax71] J. Ax. On schanuel's conjectures. *Annals of Mathematics*, 93(2):252–268, 1971.

[Ax72] J. Ax. Some topics in differential algebraic geometry i. *American Journal of Mathematics*, 94(4):1195–1204, 1972.

[BNZ22] G. Binyamini, D. Novikov, and B. Zack. Wilkie's conjecture for pfaffian structures. Available at: https://arxiv.org/abs/2202.05305, 2022.

[Boo86] W. M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry; 2nd ed.* Pure and applied mathematics (Elsevier). Academic Press, Orlando, FL, 1986.

[BvdD22] N. Bhardwaj and L. van den Dries. On the pila-wilkie theorem. *Expositiones Mathematicae*, 2022.

[Cos99] M. Coste. An introduction to o-minimal geometry. Available at: https://perso.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf, 1999.

[Cos02] M. Coste. An introduction to semialgebraic geometry. Available at: https://gcomte.perso.math.cnrs.fr/M2/CosteIntroToSemialGeo.pdf, 2002.

[FG02] K. Fritzsche and H. Grauert. *From Holomorphic Functions to Complex Manifolds*. Graduate Texts in Mathematics. Springer New York, 2002.

[Gab68] A. Gabrielov. Projections of semi-analytic sets. *Functional Anal. Appl.*, 2, 1968.

[Hab15] P. Habegger. The manin-mumford conjecture, an elliptic curve, its torsion points their galois orbits. *LMS Lecture Note Series*, 421:1–40, 2015.

[Hum75] J.E. Humphreys. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, 1975.

[Lan02] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York / Berlin / Heidelberg, 2002.

[Lin14] N. Lindner. The dual abelian variety. Available at: https://www2.mathematik.hu-berlin.de/~bakkerbe/Abelian9.pdf, 2014.

[Łoj65] S. Łojasiewicz. *Ensembles semi-analytiques*. Institut des Hautes Etudes Scientifiques, 1965.

[Man65] H. B. Mann. On linear relations between roots of unity. *Mathematika*, 12(2):107–117, 1965.

[Man20] F. Mangolte. *Real algebraic varieties*. Springer Nature Switzerland, 2020.

[Mar02] D. Marker. *Model Theory: An introduction*. Springer New York, NY, 2002.

[Mar10] D. Marker. Model theory for algebra and algebraic geometry. Available at: http://homepages.math.uic.edu/~marker/orsay/mm.pdf, 2010.

[Mas84] D.W. Masser. Small values of the quadratic part of the Néron-Tate height on an abelian variety. *Compositio Mathematica*, 53(2):153–170, 1984.

[Mil08] J. Milne. Abelian varieties. Available at: https://www.jmilne.org/math/, 2008.

[Mil12] J. Milne. Algebraic geometry (v5.22). Available at: www.jmilne.org/math/, 2012.

[Mil17] J. Milne. Algebraic geometry (v6.02). Available at: https://www.jmilne.org/math/, 2017.

[Mum70] D. Mumford. *Abelian Varieties.* Oxford University Press, 1970.

[Mum76] D. Mumford. *Algebraic Geometry I: Complex Projective Varieties.* Grundlehren der mathematischen Wissenschaften. Springer Berlin / Heidelberg / New York, 1976.

[Mum88] D. Mumford. *The Red Book of Varieties and Schemes.* Lecture Notes in Mathematics. Springer Berlin / Heidelberg, 1988.

[MW96] A. Macintyre and A.J. Wilkie. On the decidability of the real exponential field. In Piergiorgio Odifreddi, editor, *Kreiseliana. About and Around Georg Kreisel*, pages 441–467. A K Peters, 1996.

[Orr15] M. Orr. Introduction to abelian varieties and the ax-lindemann-weierstrass theorem. *LMS Lecture Note Series*, 421:100–128, 2015.

[Pil04] J. Pila. Integer points on the dilation of a subanalytic surface. *The Quarterly Journal of Mathematics*, 55(2):207–223, 2004.

[Pil15] J. Pila. Functional transcendence via o-minimality. *LMS Lecture Note Series*, 421:66–99, 2015.

[PST21] J. Pila, A. N. Shankar, and J. Tsimerman. Canonical heights on shimura varieties and the andré-oort conjecture. Available at: https://arxiv.org/abs/2109.08788, 2021.

[PT13] J. Pila and J. Tsimerman. The andré-oort conjecture for the moduli space of abelian surfaces. *Compositio Math.*, 149(2):204–216, 2013.

[PW06] J. Pila and A. J. Wilkie. The rational points of a definable set. *Duke Mathematical Journal*, 133(3):591 – 616, 2006.

[PZ08] J. Pila and U. Zannier. Rational points in periodic analytic sets and the manin-mumford conjecture. *Rend. Lincei (9) Mat. Appl.*, 19(2):149–162, 2008.

[R+78] S. Ramanan et al. *A note on C.P. Ramanujam, in: C.P. Ramanujam - A Tribute.* Springer Berlin / Heidelberg / New York, 1978.

[Ray83] M. Raynaud. Sous-variétés d'une variété abélienne er points de torsion. *Arithmetic and Geometry*, 1, 1983.

[Sca12] T. Scanlon. Theorems on unlikely intersections by counting points in definable sets. Available at: http://zilberpink-2012.institut.math.jussieu.fr/Scanlon.pdf, 2012.

[Sca17] T. Scanlon. O-minimality as an approach to the andré-oort conjecture. Available at: https://math.berkeley.edu/~scanlon/papers/omaomar16.pdf, 2017.

[Ser55] J.-P. Serre. Faisceaux algébriques cohérents. *Annals of Mathematics*, 61(2):197–278, 1955.

[Ser56] J.-P. Serre. Géométrie algébrique et géométrie analytique. *Annales de l'institut Fourier*, 6:1–42, 1956.

[Sil09] J. H. Silverman. *The Arithmetic of Elliptic Curves*. Graduate texts in mathematics. Springer, Dordrecht, 2009.

[Tar49] A. Tarski. A decision method for elementary algebra and geometry. *Journal of Symbolic Logic*, 14(3):188–188, 1949.

[vdD82] L. van den Dries. Some applications of a model theoretic fact to (semi-) algebraic geometry. *Indagationes Mathematicae (Proceedings)*, 85(4):397–401, 1982.

[vdD86] L. van den Dries. A generalization of the tarski-seidenberg theorem, and some nondefinability results. *Bulletin of the American Mathematical Society*, 15(2):189–193, October 1986.

[vdD98] L. van den Dries. *Tame Topology and O-minimal Structures*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998.

[vdDM94] L. van den Dries and C. Miller. On the real exponential field with restricted analytic functions. *Israel Journal of Mathematics*, 85:19–56, 1994.

[vdDMM94] L. van den Dries, A. Macintyre, and D. Marker. The elementary theory of restricted analytic fields with exponentiation. *Annals of Mathematics*, 140(1):183–205, 1994.

[Wer11] K. Werndli. Elementary gaga. Master's thesis, 2011. Available at: https://sma.epfl.ch/~werndli/scripts/mscthesis/elementary_gaga.pdf.

[Wil96] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential function. *Journal of the American Mathematical Society*, 9:1051–1094, 1996.