ARITHMETISCHE GEOMETRIE OBERSEMINAR

Shtuka and the global Langlands correspondence, after V. Lafforgue

PROGRAMMVORSCHLAG: MICHAEL RAPOPORT UND PETER SCHOLZE

SOMMERSEMESTER 2014

In this ARGOS we want to study the paper [VL] by V. Lafforgue where he proves the global Langlands conjecture for an arbitrary reductive group G over a function field over a finite field, in the direction from automorphic representations to Galois representations. In particular, this reproves L. Lafforgue's results, [LL], in the case of $G = \mathrm{GL}_n$.
V. Lafforgue's arguments are very geometric, and rely on the geometric Satake equivalence of Mirković and Vilonen, [MV]. We will start by going through T. Richarz's paper on the geometric Satake equivalence, [R], and the definition of shtukas and the analogue of the local model diagram in that setup, [VL, Section 2].

1) Affine Grassmannians and Beilinson-Drinfeld Grassmannians
Define the affine Grassmannian, the Beilinson-Drinfeld Grassmannian and the convolution morphism, following [R, Section 2.1].

2) Perverse Sheaves on the affine Grassmannian
Introduce the 'Satake category' of equivariant perverse sheaves on the affine Grassmannian. Show that it is semisimple, and describe the simple objects. (Cf. [R, Section 3].) Provide some background on perverse sheaves as needed.

3) The fusion product
Compare the convolution product of perverse sheaves with the fusion product. For this, define ULA sheaves and state their basic properties. Use this to give the Satake category a symmetric monoidal structure, and show that the associated Tannakian group $\check{G}$ is reductive.[1] (Cf. [R, Section 2.2].)

4) Shtuka and the local model diagram
For simplicity, we will restrict ourselves to the case $G = \mathrm{GL}_n$. Define the moduli space of shtuka in the GL_n-case, and prove the 'local model diagram', [VL, Section 2, Proposition 2.8]. Cover the rest of [VL2, Section 3]: Define the sheaves $\mathcal{H}_{I,W}$ and use the geometric Satake equivalence to prove [VL2, Théorème 3.8].

5) Drinfeld's lemma, and the Hecke-finite part
Prove the lemma of Drinfeld, [VL2, Lemme 4.1] (cf. the references given there). Define the Hecke-finite part, and prove [VL2, Proposition 1.3] modulo some auxiliary results, as in [VL2, Section 4].

6) Excursion operators, and pseudorepresentations
Prove the spectral decomposition according to Langlands parameters, [VL2, Théorème 1.1]. For this, define the excursion operators, state their basic properties with some indication of the proof, and then construct the desired Langlands parameters by some form of pseudorepresentations, following [VL2, Section 5]. It may be useful to restrict to $G = \mathrm{GL}_n$ here (cf. [VL2, p. 31, l.3–6]).

[1] For the seminar, we will use this as the *definition* of $\check{G}$; we will not need the identification of $\check{G}$ with the Langlands dual group.

7) Excursion operators and Hecke operators, I
The remaining key result is a comparison of excursion operators and Hecke operators, given by [VL2, Proposition 6.3]. This result will be proved in the next talk. In this talk, set the stage for the argument by covering the rest of [VL2, Section 6], including the congruence relation [VL2, Proposition 6.4] as a consequence.

8) Excursion operators and Hecke operators, II
Give the proof of [VL2, Proposition 6.3], at least in the case where V is minuscule and $\deg(v) = 1$. If time permits, explain the strategy in the general case.

9) Conclusion
Finish the proof by showing that the decomposition given in talk 6 is compatible with the unramified local Langlands correspondence, as in [VL2, Section 9]. If time permits, explain parts of [VL2, Sections 7,8], where the auxiliary results used in talk 5 are proved.

References

[LL] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. 147 (2002), no. 1, 1–241.

[VL] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, arXiv:1209.5352v4.

[VL2] V. Lafforgue, *Introduction aux chtoucas pour les groupes réductifs et à la paramétrisation de Langlands globale*, arXiv:1404.3998v1.

[MV] Mirković, I., Vilonen, K., *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[R] Richarz, T., *A new approach to the geometric Satake equivalence*, Doc. Math. 19 (2014), 209–246.