ARithmetische Geometrie OberSeminar

Bonn, Wintersemester 2016/17

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The stack of vector bundles on the Fargues–Fontaine curve, and sheaves on it

In the last ARGOS, we went through foundations on the étale cohomology of diamonds. This semester, we will apply the theory to the study of the stack of vector bundles on the Fargues–Fontaine curve. The main theorem is the following, where we have fixed a nonarchimedean local field F of residue characteristic p, an integer $n \geq 1$, and $\ell \neq p$.

Theorem 1. *Let* Bun_n *be the stack of rank* n *vector bundles on the Fargues–Fontaine curve. Then* Bun_n *is a "smooth Artin stack of dimension* 0*", and* $\mathcal{F} \in D_{\text{ét}}(\mathrm{Bun}_n, \mathbb{Z}/\ell^m\mathbb{Z})$ *is reflexive if and only if (all cohomology groups of) all stalks of* $\mathcal{F}$ *are admissible representations of the automorphism group of the point.*

The theorem is valid for any reductive group G, but to avoid group-theoretic difficulties and concentrate on geometric difficulties, we will restrict to $G = \mathrm{GL}_n$.

Talks

1. Talk: The Fargues–Fontaine curve

Define the (relative) Fargues–Fontaine curve over any perfectoid space of characteristic p. In the case of a geometric point, discuss its schematic version, and the classification of vector bundles on it. Moreover, describe vector bundles on the Fargues–Fontaine curve in terms of φ-modules on the Robba ring. References: [4, Sections 6 and 8.7], [2, Chapters 6 and 8]

2. Talk: Families of vector bundles, I

Prove that for a vector bundle on the relative Fargues–Fontaine, the Harder–Narasimhan polygon is semicontinuous. In particular, the semistable locus is open. Reference: [4, Section 7.4]

3. Talk: Families of vector bundles, II

Prove that a semistable vector bundle on the relative Fargues–Fontaine curve becomes trivial after a pro-étale cover. Reference: [4, Section 7.3]

4. Talk: Banach–Colmez spaces

Define the category of Banach–Colmez spaces, and show that they are smooth. Show that if $\mathcal{E}$ is a vector bundle on the relative Fargues–Fontaine curve which has everywhere only positive slopes, then the global sections of $\mathcal{E}$ define a smooth morphism to the base. Moreover, describe the automorphism groups of non-semistable vector bundles. References: [1], [3]

5. Talk: Sheaves on $[*/\underline{G(\mathbb{F})}]$

Show that for a linear group G over $\mathbb{F}$, $[*/\underline{G(\mathbb{F})}]$ is a smooth Artin stack. Identify $D_{\text{ét}}([*/\underline{G(\mathbb{F})}], \Lambda)$ with the derived category of smooth representations of $G(\mathbb{F})$, and Verdier duality with smooth duality. Reference: [3]

6. Talk: Strategy of proof of main theorem

Explain the strategy of the proof of the main theorem. In particular, for any $b \in B(\mathrm{GL}_n)$, define the local chart $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_n$ of Bun_n. Reference: [3]

7. Talk: Smoothness of π_b

Describe the $J_b(F)$-torsor $\widetilde{\mathcal{M}}_b$ over $\mathcal{M}_b$ explicitly in terms of Banach–Colmez spaces, and in particular explain in detail the geometry of the first nontrivial example for GL_2. Moreover, prove that the map $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_n$ is smooth. Reference: [3]

8. Talk: Finiteness of cohomology of $\widetilde{\mathcal{M}}_b^\circ$

Prove a finiteness result for the cohomology of the open part $\widetilde{\mathcal{M}}_b^\circ \subset \widetilde{\mathcal{M}}_b$. Reference: [3]

9. Talk: Local Poincaré duality

Prove local Poincaré duality on the space $\widetilde{\mathcal{M}}_b^\circ$ by reducing it to Poincaré duality on the proper and smooth space $\widetilde{\mathcal{M}}_b^\circ/U_p^{\mathbb{Z}}$. Reference: [3]

10. Talk: Conclusion, and applications

Put everything together to prove the main theorem. Deduce finiteness results for the cohomology of Rapoport–Zink spaces. Explain that this requires invariance of reflexive sheaves on Bun_n under base change to an algebraically closed nonarchimedean field, and prove this. Reference: [3]

11. Talk: Hecke operators

Show how a geometric Satake equivalence for the B_{dR}^+-affine Grassmannian would yield an action of the category of algebraic representations of (the Langlands dual) GL_n on the category of reflexive sheaves on Bun_n; construct this action unconditionally for GL_n. Explain how this yields a construction of local Langlands parameters, and compare it with the known construction via the cohomology of the Lubin–Tate tower. Reference: [3]

References

[1] P. Colmez, *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu 1, No. 3, 331–439 (2002).

[2] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, https://webusers.imj-prg.fr/~laurent.fargues/Courbe_fichier_principal.pdf

[3] L. Fargues, P. Scholze, *Geometrization of the local Langlands correspondence*, in preparation.

[4] K. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, Astérisque 371.