Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 2

Due Date: 05.05.2014

Exercise 1:

(i) Show that a morphism $f : X \to Y$ is a monomorphism (i.e. $f \circ g = f \circ h \Rightarrow g = h$ for morphisms $g, h : T \to X$) if and only if the diagonal $\Delta_f : X \to X \times_Y X$ is an isomorphism.

(ii) Let $f, g : X \to Y$ be morphisms of S-schemes and assume that X is reduced and that Y is separated over S. Assume that there is a dense open subscheme $U \subset X$ such that $f|_U = g|_U$. Show that $f = g$.

(iii) Let $f : X \to Y$ be a separated morphism. Let $g : Y \to X$ be a section of f, i.e. a morphism such that $f \circ g = \mathrm{id}_Y$. Show that g is a closed immersion.

Exercise 2:

Let X be a scheme and let $\mathscr{A} = \bigoplus_{d \geq 0} \mathscr{A}_d$ be a quasi-coherent graded $\mathcal{O}_X$-algebra.

(i) Show that there is an X-scheme $\pi : \underline{\mathrm{Proj}}_X \mathscr{A} \to X$ such that for all affine open subschemes $U \subset X$ there is an isomorphism $\varphi_U : \pi^{-1}(U) \cong \mathrm{Proj}\,(\Gamma(U, \mathscr{A}))$ of U-schemes, and for all affine open subschemes $V \subset U$ the diagram

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\varphi_V} & \mathrm{Proj}\,(\Gamma(V, \mathscr{A})) \\ \downarrow & & \downarrow \\ \pi^{-1}(U) & \xrightarrow{\varphi_U} & \mathrm{Proj}\,(\Gamma(U, \mathscr{A})) \end{array}$$

commutes. Here the vertical arrow on the right is induced by the restriction map

$$\Gamma(U, \mathscr{A}) \longrightarrow \Gamma(V, \mathscr{A}).$$

(ii) Let $\mathscr{L}$ be a line bundle on X and define $\mathscr{A}' = \bigoplus_{d \geq 0} \mathscr{A}_d \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes d}$ which has a canonical structure as an $\mathcal{O}_X$-algebra. Show that there is a canonical isomorphism of X-schemes

$$\underline{\mathrm{Proj}}_X \mathscr{A} \cong \underline{\mathrm{Proj}}_X \mathscr{A}'.$$

(iii) Assume that X is noetherian that $\mathscr{A}_0 = \mathcal{O}_X$ and $\mathscr{A}_1$ is coherent and that $\mathscr{A}$ is generated (as an $\mathcal{O}_X$-algebra) by $\mathscr{A}_1$. Show that π is proper.

Exercise 3:
A morphism $f : X \to Y$ of schemes is called *projective* if there is a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}^n_Y \\ & \searrow^{f} & \downarrow \\ & & Y \end{array}$$

where i is a closed immersion.

(i) Show that the base change of a projective morphism is projective.

(ii) Show that the composition of projective morphisms is projective.

(*Hint: Use the Segre-embedding*)

Exercise 4:

(i) Let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of rank d on scheme X. Show that $\mathbf{P}(\mathscr{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^\bullet \mathscr{E})$ is locally on X isomorphic to $\mathbb{P}^{d-1}_X = X \times \mathbb{P}^{d-1}_{\mathbb{Z}}$.

(ii) Assume that X is affine. Show that $\mathbf{P}(\mathscr{E}) \to X$ is projective in the sense of exercise 3.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`