Exercises for **Topology I** Sheet 7

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Definition. Let \mathcal{D} be a small category. The *nerve of* \mathcal{D} is the simplicial set $N(\mathcal{D})$ with

$$N(\mathcal{D})_n = Hom_{Cat}(\mathcal{C}_{[n]}, \mathcal{D})$$

and structure map $\alpha^* \colon \mathcal{N}(\mathcal{D})_n \to \mathcal{N}(\mathcal{D})_m$ for $[m] \to [n]$ given by precomposition with $\mathcal{C}_{\alpha} \colon \mathcal{C}_{[m]} \to \mathcal{C}_{[n]}$. Here $\mathcal{C}_{(-)} \colon \mathbf{Poset} \to \mathbf{Cat}$ is as on the previous sheet.

- **Exercise 1.** 1. Let $n \ge 0$. Construct a bijection between $N(\mathcal{D})_n$ and the set N_n of tuples $(\alpha_n, \ldots, \alpha_1)$ of composable arrows in \mathcal{D} (i.e. such that the source of α_{i+1} equals the target of α_i); by convention, N_0 is the set of objects of \mathcal{D} . What do the face and degeneracy maps d_i^* , s_i^* correspond to under this identification?
 - 2. Upgrade N to a fully faithful functor $Cat \rightarrow SSet$ from the category of small categories to the category of simplicial sets.
- *3. (3 bonus points) Let X be a simplicial set such that every morphism $\Lambda_k^n \to X$ with $0 < k < n \leq 3$ admits a unique extension to Δ^n . Construct a small category hX with objects X_0 and with hom sets $\operatorname{Hom}_{hX}(x,y) \coloneqq \{e \in X_1 : d_1^*(e) = x, d_0^*(e) = y\}.$
- *4. (7 bonus points) Show: a simplicial set X is isomorphic to the nerve of a small category if and only if each $\Lambda_k^n \to X$ with 0 < k < n admits a unique extension to Δ^n .
- **Exercise 2.** 1. For a family $(C^i)_{i \in I}$ of chain complexes we define the direct sum $\bigoplus_{i \in I} C^i$ dimensionwise, i.e. $(\bigoplus_{i \in I} C^i)_n = \bigoplus_{i \in I} C^i_n$ with componentwise differential.

Show that the inclusions $C_n^i \hookrightarrow \bigoplus_{i \in I} C_n^i$ define morphisms of chain complexes $\operatorname{incl}_i : C^i \to \bigoplus_{i \in I} C^i$, and that they have the following universal property: for every chain complex D the map

$$\operatorname{Hom}\left(\bigoplus_{i\in I} C^{i}, D\right) \longrightarrow \prod_{i\in I} \operatorname{Hom}(C^{i}, D)$$
$$f \longmapsto (f \circ \operatorname{incl}_{i})_{i\in I}$$

is bijective.

- 2. Show that the homomorphism $\bigoplus_{i \in I} H_n(C^i) \to H_n(\bigoplus_{i \in I} C^i)$ induced by the inclusions $C^i \to \bigoplus_{i \in I} C^i$ is an isomorphism of abelian groups for every $n \ge 0$.
- 3. Let $\{Y_i\}_{i \in I}$ be a family of simplicial sets and let A be an abelian group. Show that the homomorphism $\bigoplus_{i \in I} C(Y_i, A) \to C(\coprod_{i \in I} Y_i, A)$ induced by the inclusions is an isomorphism of chain complexes.
- 4. Let X be a topological space and let $(U_i)_{i \in I}$ be pairwise disjoint open subsets of X with $X = \bigcup_{i \in I} U_i$. Construct an isomorphism of simplicial sets $\coprod_{i \in I} \mathcal{S}(U_i) \to \mathcal{S}(X)$ and use this to construct isomorphisms $\bigoplus_{i \in I} H_n(U_i, A) \cong H_n(X, A)$ for all $n \ge 0$ and all abelian groups A.

please turn over

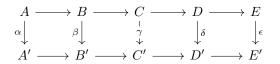
- **Exercise 3.** 1. Let X be a simplicial set and let $Y \subseteq X$ be a subsimplicial set. Show that there is a unique way to turn the degreewise quotients $(X/Y)_n := X_n/Y_n$ into a simplicial set X/Y in such a way that the quotient maps form a morphism of simplicial sets $X \to X/Y$.
 - 2. For $n \ge 1$ we define the *boundary* $\partial \Delta^n \subseteq \Delta^n$ as the subsimplicial set given by the non-surjective maps (you can convince yourself that this indeed a subsimplicial set). Show that

$$H_m(\Delta^n/\partial\Delta^n, A) \cong \begin{cases} A & \text{if } m = 0 \text{ or } m = n \\ 0 & \text{if } 0 < m < n \end{cases}$$

for every abelian group A, and make the isomorphisms in degrees 0 and n explicit.

Remark. The quotient $\Delta^n/\partial\Delta^n$ is sometimes called the (standard) simplicial n-sphere. We will later see that $H_m(\Delta^n/\partial\Delta^n, A) = 0$ for m > n.

Exercise 4. Let



be a commutative diagram of groups and homomorphisms such that both rows are exact. Show the following:

- 1. If β and δ are injective and α is surjective, then γ is injective.
- 2. If β and δ are surjective and ϵ is injective, then γ is surjective.
- 3. If β and δ are bijective, α is surjective, and ϵ is injective, then γ is bijective.