CATEGORIFICATION OF A LINEAR ALGEBRA IDENTITY AND FACTORIZATION OF SERRE FUNCTORS

SEFI LADKANI

Abstract. We provide a categorical interpretation of a well-known identity from linear algebra as an isomorphism of certain functors between triangulated categories arising from finite dimensional algebras.

As a consequence, we deduce that the Serre functor of a finite dimensional triangular algebra A has always a lift, up to shift, to a product of suitably defined reflection functors in the category of perfect complexes over the trivial extension algebra of A.

1. Introduction

The general philosophy behind categorification, as explained for example in [2], is that numbers should be interpreted as sets, sets as categories, equalities as isomorphisms and so on. When one considers linear operators, the following suggested interpretation makes sense, see also [15] for a similar definition.

Given the data of a free $\mathbb{Z}$-module V of finite rank and linear maps $f_1, f_2, \dots, f_n, g : V \to V$ satisfying $g = f_1 \cdot f_2 \cdot \dots \cdot f_n$, a (weak) *categorification* of this data consists of an abelian or triangulated category $\mathcal{B}$ whose Grothendieck group $K_0(\mathcal{B})$ is isomorphic to V, together with exact functors $F_i : \mathcal{B} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{B}$, such that:

- $F_1, F_2, \dots, F_n, G$ induce linear maps on $K_0(\mathcal{B})$ which, under the isomorphism with V, coincide with $f_1, f_2, \dots, f_n, g$;
- There is an isomorphism of functors between G and the composition $F_1 \cdot F_2 \cdot \dots \cdot F_n$.

When V carries additional structure, such as a bilinear form, it is preferable that this structure lifts to $\mathcal{B}$ as well.

1.1. A linear algebra identity.

The following well-known statement concerns products of reflection-like matrices defined by a square matrix.

Proposition 1.1. *Let B be any square $n \times n$ matrix over a commutative ring. Then*

$$-B_+^{-1} B_-^T = r_1^B \cdot r_2^B \cdot \dots \cdot r_n^B, \tag{1.1}$$

where the matrices B_+ and B_- are the upper and lower triangular parts of B, defined by

$$(B_+)_{ij} = \begin{cases} B_{ij} & \text{if } i < j, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad (B_-)_{ij} = \begin{cases} B_{ji} & \text{if } i < j, \\ B_{ii} - 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

(so that $B = B_+ + B_-^T$), and for each $1 \leq i \leq n$, the square matrix r_i^B is obtained from the identity matrix by subtracting the i-th row of B, that is,

$$\left(r_i^B\right)_{st} = \delta_{st} - \delta_{si} B_{it}, \qquad 1 \leq s,t \leq n. \tag{1.2}$$

This statement originally appeared as an exercise in the book of Bourbaki [5, Ch. 5, §6, no. 3], following an argument presented in Coxeter's paper [9]. Various specific cases have since then appeared in the literature, including A'Campo [1] in the bipartite case and Howlett [14] in the symmetric case. The general form is stated and proved in an article by Coleman [8], and an alternative proof can be found in [16].

As important special case is when $B = C + C^T$ is the symmetrization of an upper triangular square matrix C with ones on its main diagonal. In this case the matrices r_i^B are reflections, and the proposition implies that

$$-C^{-1}C^T = r_1^B \cdot r_2^B \cdot \ldots \cdot r_n^B. \tag{1.3}$$

This equality provides us with two points of view on the so-called *Coxeter transformation*. First, as known in Lie theory, it is the product of the simple reflections, as given by the right hand side of (1.3). Second, as follows from the left hand side, it can also be described as the automorphism Φ satisfying

$$\langle x, y \rangle_C = - \langle y, \Phi x \rangle_C$$

where $\langle \cdot, \cdot \rangle_C$ is the bilinear form defined by the matrix C and x, y are any two vectors, as known in the representation theory of algebras, see [17].

1.2. **Categorical interpretation.** Our categorical interpretation of equations (1.1) and (1.3) is achieved by using functors on triangulated categories arising from finite dimensional algebras. In order to state our result in precise terms, we need to recall a few notions from the representation theory of finite dimensional algebras.

For a finite dimensional algebra A over a field k, denote by $\mathcal{D}^b(A)$ the bounded derived category of finite dimensional right A-modules, and by $\operatorname{per} A$ its full triangulated subcategory consisting of all complexes quasi-isomorphic to perfect complexes, that is, bounded complexes whose terms are finitely generated projective A-modules.

The Grothendieck group $K_0(\operatorname{per} A)$ is free abelian of finite rank, with a basis consisting of the classes of the indecomposable projective A-modules. It is equipped with a bilinear form induced by the Euler form

$$\langle X, Y \rangle_A = \sum_{r \in \mathbb{Z}} (-1)^r \dim_k \operatorname{Hom}_{\mathcal{D}^b(A)}(X, Y[r]) \qquad X, Y \in \operatorname{per} A.$$

The algebra A is called *triangular* if there exist primitive orthogonal idempotents $e_1, \ldots, e_n$ of A such that $e_i A e_j = 0$ for any $j < i$ and $e_i A e_i \simeq k$ for $1 \leq i \leq n$. The modules $P_i = e_i A$ then form a complete collection of indecomposable projectives. Taking their classes as a basis for $K_0(\operatorname{per} A)$, it will be convenient for us to order them $[P_n], \ldots, [P_1]$ and to define the *Cartan matrix* C_A as the matrix of $\langle \cdot, \cdot \rangle_A$ with respect to that basis, namely

$$\begin{aligned}(C_A)_{ij} &= \langle P_{n+1-i}, P_{n+1-j} \rangle_A = \dim_k \operatorname{Hom}_A(P_{n+1-i}, P_{n+1-j}) \\ &= \dim_k e_{n+1-j} A e_{n+1-i},\end{aligned}$$

so that C_A is upper triangular with ones on its main diagonal.

Similarly, for a (finite dimensional) A-A-bimodule M we can define a matrix C_M by

$$(C_M)_{ij} = \dim_k e_{n+1-j} M e_{n+1-i},$$

and call M *triangular* if C_M is upper triangular, or equivalently, $e_i M e_j = 0$ for any $j < i$. We have $C_M^T = C_{DM}$, where DM is the *dual* of M, defined as $DM = \operatorname{Hom}_k(M, k)$.

The *trivial extension* $\Lambda = A \ltimes DM$ is the k-algebra which has $A \oplus DM$ as its underlying vector space, with the multiplication defined by $(a, \mu)(a', \mu') = (aa', a\mu' + \mu a')$. Its indecomposable projectives are in bijective correspondence with those of A, and its Cartan matrix is given by $C_\Lambda = C_A + C_M^T$. Thus, when A and M are triangular, $(C_\Lambda)_+ = C_A$ and $(C_\Lambda)_- = C_M$.

Theorem 1.2. *Let A be a finite dimensional triangular algebra over a field and let ${}_A M_A$ be a triangular A-A-bimodule. Set $\Lambda = A \ltimes DM$ to be the trivial extension of A with the dual of M.*

Then there exist, for $1 \le i \le n = \operatorname{rank} K_0(\operatorname{per} \Lambda)$, triangulated functors $R_i : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda)$ which restrict to $R_i : \operatorname{per} \Lambda \to \operatorname{per} \Lambda$, such that:

(a) *Each functor R_i induces a linear map on $K_0(\operatorname{per} \Lambda)$ whose matrix with respect to the basis of indecomposable projective Λ-modules is $r_{n+1-i}^{C_\Lambda}$, cf. (1.2), where C_Λ is the Cartan matrix of Λ;*

(b) *The diagrams of triangulated functors*

$$\begin{array}{ccccccc} \operatorname{per}\Lambda & \xrightarrow{R_1} & \operatorname{per}\Lambda & \xrightarrow{R_2} & \cdots & \xrightarrow{R_n} & \operatorname{per}\Lambda \\ \downarrow {\scriptstyle -\overset{\mathbf{L}}{\otimes}_\Lambda A_A} & & & & & & \downarrow {\scriptstyle -\overset{\mathbf{L}}{\otimes}_\Lambda A_A} \\ \mathcal{D}^b(A) & & & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A DM_A[1]} & & & \mathcal{D}^b(A) \end{array} \tag{1.4}$$

and

$$\begin{array}{ccccccc} \mathcal{D}^b(A) & & & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A DM_A[1]} & & & \mathcal{D}^b(A) \\ \downarrow {\scriptstyle -\otimes_A A_\Lambda} & & & & & & \downarrow {\scriptstyle -\otimes_A A_\Lambda} \\ \mathcal{D}^b(\Lambda) & \xrightarrow{R_1} & \mathcal{D}^b(\Lambda) & \xrightarrow{R_2} & \cdots & \xrightarrow{R_n} & \mathcal{D}^b(\Lambda) \end{array}$$

commute up to a natural isomorphism of functors.

The vertical arrows of (1.4) induce an isomorphism $K_0(\operatorname{per} \Lambda) \to K_0(A)$ sending projectives to projectives. Thus, by considering the diagram (1.4) at the level of the Grothendieck groups, we get the following commutative diagram

$$\begin{array}{ccccccc} K_0(\operatorname{per}\Lambda) & \xrightarrow{r_n^{C_\Lambda}} & K_0(\operatorname{per}\Lambda) & \xrightarrow{r_{n-1}^{C_\Lambda}} & \cdots & \xrightarrow{r_1^{C_\Lambda}} & K_0(\operatorname{per}\Lambda) \\ \downarrow {\scriptstyle I_n} & & & & & & \downarrow {\scriptstyle I_n} \\ K_0(A) & & & \xrightarrow{-(C_\Lambda)_+^{-1}(C_\Lambda)_-^T} & & & K_0(A) \end{array}$$

(where I_n is the $n \times n$ identity matrix), which explains why the theorem can be seen as a categorical interpretation of (1.1) for $B = C_\Lambda$.

To complement this result, we note that any integral square matrix B with non-negative entries and positive entries on its main diagonal can be realized as a Cartan matrix of a suitable Λ as in the theorem. More precisely, there exist a finite dimensional triangular algebra A and a triangular bimodule M over A with $B_+ = C_A$ and $B_- = C_M$, see Section 2.7 for the details.

1.3. **Application to Serre functors.** A triangular finite dimensional algebra A is of finite global dimension, hence its bounded derived category $\mathcal{D}^b(A)$ admits a Serre functor ν_A in the sense of Bondal and Kapranov [4]. By a result of Happel [12], it is given by the left derived functor of the Nakayama functor, $\nu_A = - \overset{\mathbf{L}}{\otimes}_A DA$. Thus, by taking in Theorem 1.2 the bimodule M to be A, we deduce the following result on the Serre functor on $\mathcal{D}^b(A)$.

Corollary 1.3. *Let A be a finite dimensional triangular algebra over a field and let $T(A) = A \ltimes DA$ be its trivial extension algebra. Then there exist, for $1 \leq i \leq n = \operatorname{rank} K_0(A)$, triangulated autoequivalences R_i on $\mathcal{D}^b(T(A))$ which restrict to autoequivalences on* $\operatorname{per} T(A)$*, such that:*

(a) *Each autoequivalence R_i induces a linear map on $K_0(\operatorname{per} T(A))$, whose matrix with respect to the basis of indecomposable projective $T(A)$-modules is given by the reflection r^B_{n+1-i}, where B is the symmetrization of the Cartan matrix of A;*

(b) *The diagrams of triangulated functors*

$$\begin{array}{ccccccc} \operatorname{per} T(A) & \xrightarrow{R_1} & \operatorname{per} T(A) & \xrightarrow{R_2} & \cdots & \xrightarrow{R_n} & \operatorname{per} T(A) \\ \downarrow {\scriptstyle -\overset{\mathbf{L}}{\otimes}_{T(A)} A_A} & & & & & & \downarrow {\scriptstyle -\overset{\mathbf{L}}{\otimes}_{T(A)} A_A} \\ \mathcal{D}^b(A) & & & \xrightarrow{\nu_A[1]} & & & \mathcal{D}^b(A) \end{array} \tag{1.5}$$

and

$$\begin{array}{ccccccc} \mathcal{D}^b(A) & & & \xrightarrow{\nu_A[1]} & & & \mathcal{D}^b(A) \\ \downarrow {\scriptstyle -\otimes_A A_{T(A)}} & & & & & & \downarrow {\scriptstyle -\otimes_A A_{T(A)}} \\ \mathcal{D}^b(T(A)) & \xrightarrow{R_1} & \mathcal{D}^b(T(A)) & \xrightarrow{R_2} & \cdots & \xrightarrow{R_n} & \mathcal{D}^b(T(A)) \end{array}$$

commute up to a natural isomorphism of functors.

Thus, one can lift (a shift of) the Serre functor on $\mathcal{D}^b(A)$ to a product of the "reflections" R_i in $\operatorname{per} T(A)$. As before, the diagram (1.5) can be regarded as a categorical interpretation of equation (1.3) for $C = C_A$, the Cartan matrix of A. This can be done for any upper triangular integral matrix C with non-negative entries and 1 on its main diagonal, see Section 2.7.

1.4. **On the proof.** Section 2 is devoted to the proof of the theorem and its corollaries. A key ingredient in the proof is the proper definition and analysis of the functors R_i. They are defined, for each $1 \leq i \leq n$, as the left derived functors of tensoring with a two-term complex of bimodules,

$$R_i^\Lambda = - \overset{\mathbf{L}}{\otimes}_\Lambda \left(\Lambda e_i \otimes_k e_i \Lambda \xrightarrow{m} \Lambda \right)$$

where m denotes the multiplication map and $e_1, \dots, e_n$ are the primitive orthogonal idempotents.

The functors R_i^Λ have already been considered in the works of Rouquier-Zimmermann [19] on braid group actions on derived categories of Brauer tree algebras without exceptional vertex, and by Hoshino and Kato [13] in relation with constructions of two-sided tilting complexes for self-injective algebras. When the algebra Λ is symmetric and $\dim e_i \Lambda e_i = 2$, the functor R_i^Λ can be viewed as a twist functor in the sense of Seidel and Thomas [20] with respect to the 0-spherical object $e_i\Lambda$. Our result shows the importance of the functors R_i^Λ for a wider class of algebras Λ, which are not necessarily restricted to be self-injective or symmetric.

In the course of the proof we establish the special case of Theorem 1.2 where the bimodule M is zero, namely that for any finite-dimensional triangular algebra A, the composition $R_n^A \cdot \ldots \cdot R_2^A \cdot R_1^A$ is isomorphic to zero on $\mathcal{D}^b(A)$, see Section 2.2.

Plugging in this statement the definition of R_i^A, we obtain a (finite) projective resolution of the triangular algebra A as a bimodule over itself. A similar construction, with relation to Hochschild cohomology computations, was presented by Cibils in [7].

1.5. **Previous work.** Another categorical interpretation of (1.3), in the realm of representation theory of quivers, is given by a result of Gabriel [10], correcting previous paper by Brenner and Butler [6].

For a quiver Q without oriented cycles, one can consider two exact autoequivalences on the bounded derived category of its path algebra. The first is the Auslander-Reiten translation, corresponding to the left hand side of (1.3), and the second is the so-called Coxeter functor, which was defined by Bernstein, Gelfand and Ponomarev [3] as a product of their reflection functors, corresponding to the right hand side of (1.3).

In [10], it is shown that for any quiver whose underlying graph is a tree, or more generally, does not contain a cycle of odd length, the Auslander-Reiten translation is isomorphic to the Coxeter functor, thus interpreting the equality in (1.3) as an isomorphism of functors.

In Section 3 we explain this result in more detail and compare it with our approach.

Another approach to the factorization of Serre functors for certain finite dimensional algebras, including ones arising from category $\mathcal{O}$ associated to semi-simple complex Lie algebras, is presented by Mazorchuk and Stroppel in [18].

Acknowledgements. The results in this paper were first presented at the workshop on Spectral methods in representation theory of algebras and applications to the study of rings of singularities that was held at Banff in September 2008. I would like to thank J. A. de la Peña, C. Ringel and C. Stroppel for their helpful comments and suggestions.

2. Proof of the theorem

2.1. **The building blocks – the functors R_i^Λ.** Let Λ be a basic finite dimensional algebra over a field k and let $P_1, \ldots, P_n$ be a complete collection of the non-isomorphic indecomposable projectives in $\operatorname{mod} \Lambda$, the category of finite dimensional right Λ-modules. Let $e_1, \dots, e_n$ be primitive orthogonal idempotents in Λ such that $P_i = e_i\Lambda$ for $1 \le i \le n$.

Fix $1 \le i \le n$ and consider the following complex of Λ-Λ-bimodules

$$C_i = \Lambda e_i \otimes_k e_i\Lambda \xrightarrow{m} \Lambda,$$

where Λ is in degree 0 and m is the multiplication map. Taking the tensor product $- \otimes_\Lambda C_i$ yields an endofunctor on the category $\mathcal{C}^b(\Lambda)$ of bounded complexes of finite dimensional right Λ-modules, which induces an endofunctor on its homotopy category $\mathcal{K}^b(\Lambda)$.

Since its terms are projective as left Λ-modules, the complex C_i defines a triangulated functor

$$- \overset{\mathbf{L}}{\otimes}_\Lambda C_i = - \otimes_\Lambda C_i : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda).$$

on the derived category $\mathcal{D}^b(\Lambda)$ of $\operatorname{mod}\Lambda$. Moreover, as the terms are also projective as right Λ-modules, this functor restricts to a functor

$$- \overset{\mathbf{L}}{\otimes}_\Lambda C_i = - \otimes_\Lambda C_i : \operatorname{per}\Lambda \to \operatorname{per}\Lambda$$

on the triangulated subcategory $\operatorname{per}\Lambda$ of complexes quasi-isomorphic to perfect ones (that is, bounded complexes of finitely generated projectives).

In the sequel, when no confusion arises, we shall denote all the above functors by R_i^Λ. These functors were considered by Rouquier and Zimmermann [19] in relation with braid group actions on the derived categories of Brauer tree algebras with no exceptional vertex, and by Hoshino and Kato [13] in relation with constructions of two-sided tilting complexes for self-injective algebras.

Lemma 2.1. *Let $X \in \operatorname{mod}\Lambda$. Then*

$$R_i^\Lambda(X) = \operatorname{Hom}_\Lambda(P_i, X) \otimes_k P_i \xrightarrow{ev} X$$

where ev is the evaluation map $ev : \alpha \otimes y \mapsto \alpha(y)$.

Proof. Clearly, $X \otimes_\Lambda \Lambda e_i \simeq X e_i \simeq \operatorname{Hom}_\Lambda(e_i\Lambda, X)$. □

The Grothendieck group $K_0(\operatorname{per}\Lambda)$ is a free abelian group on the generators $[P_1], \dots, [P_n]$ equipped with a bilinear form induced by the Euler form

$$\langle X, Y\rangle_\Lambda = \sum_{r\in\mathbb{Z}} (-1)^r \dim_k \operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(X, Y[r]) \qquad X, Y \in \operatorname{per}\Lambda.$$

Corollary 2.2. *Let $X \in \operatorname{per}\Lambda$. Then in $K_0(\operatorname{per}\Lambda)$ we have*

$$[R_i^\Lambda(X)] = [X] - \langle P_i, X\rangle_\Lambda [P_i].$$

Proof. Since R_i^Λ is triangulated, it is enough to verify this equality on the basis elements $[P_j]$. This follows directly from Lemma 2.1. □

The next lemma provides an explicit description of compositions of functors R_i^Λ, which will be useful in the sequel.

Lemma 2.3. *Let $s \ge 1$ and let $\varphi : \{1, \dots, s\} \to \{1, \dots, n\}$ be any function. Then*

$$R_{\varphi(s)}^\Lambda \cdot \ldots \cdot R_{\varphi(1)}^\Lambda = - \overset{\mathbf{L}}{\otimes}_\Lambda T_\varphi^\Lambda$$

for the complex T^{Λ}_{φ} *of* Λ-Λ*-bimodules given by*

$$T^{\Lambda}_{\varphi} = \dots \to 0 \to T^{\Lambda,s}_{\varphi} \xrightarrow{d^s_{\varphi}} \dots \to T^{\Lambda,r}_{\varphi} \xrightarrow{d^r_{\varphi}} T^{\Lambda,r-1}_{\varphi} \to \dots \xrightarrow{d^1_{\varphi}} T^{\Lambda,0}_{\varphi} \to 0 \to \dots$$

where

$$(2.1)\quad T^{\Lambda,0}_{\varphi} = \Lambda, \quad T^{\Lambda,r}_{\varphi} = \bigoplus_{1 \le i_1 < \dots < i_r \le s} \Lambda e_{\varphi(i_1)} \otimes e_{\varphi(i_1)} \Lambda e_{\varphi(i_2)} \otimes \dots \otimes e_{\varphi(i_r)} \Lambda$$

with the differentials d^r_{φ} *defined on each summand by*

$$(2.2)\qquad d^r_{\varphi}(\lambda_0 \otimes \lambda_1 \otimes \ldots \otimes \lambda_r) = \sum_{j=0}^{r-1} (-1)^j \lambda_0 \otimes \ldots \otimes \lambda_j \lambda_{j+1} \otimes \ldots \otimes \lambda_r$$

where $\lambda_0 \in \Lambda e_{\varphi(i_1)}$, $\lambda_r \in e_{\varphi(i_r)}\Lambda$ *and* $\lambda_j \in e_{\varphi(i_j)} \Lambda e_{\varphi(i_{j+1})}$ *for* $0 < j < r$.

Proof. By definition,

$$\begin{aligned} R^{\Lambda}_{\varphi(s)} \cdot \ldots \cdot R^{\Lambda}_{\varphi(2)} \cdot R^{\Lambda}_{\varphi(1)} &= \Big(\dots \Big(\Big(- \overset{\mathbf{L}}{\otimes}_{\Lambda} C_{\varphi(1)} \Big) \overset{\mathbf{L}}{\otimes}_{\Lambda} C_{\varphi(2)} \Big) \dots \overset{\mathbf{L}}{\otimes}_{\Lambda} C_{\varphi(s)} \Big) \\ &= - \overset{\mathbf{L}}{\otimes}_{\Lambda} \Big(C_{\varphi(1)} \otimes_{\Lambda} C_{\varphi(2)} \otimes_{\Lambda} \ldots \otimes_{\Lambda} C_{\varphi(s)} \Big) \end{aligned}$$

(where we replaced $\overset{\mathbf{L}}{\otimes}$ by $\otimes$ since the terms of C_i are projective as left (as well as right) modules), so it is enough to show that

$$T^{\Lambda}_{\varphi} = \Big(\dots \Big(C_{\varphi(1)} \otimes_{\Lambda} C_{\varphi(2)} \Big) \otimes_{\Lambda} \ldots \otimes_{\Lambda} C_{\varphi(s)} \Big)$$

where the right hand side is an iterated tensor product of complexes.

We prove this by induction on s, the case $s = 1$ being merely the definition of $R^{\Lambda}_{\varphi(1)}$. Now assume the claim for s, consider a function $\varphi : \{1, \ldots, s+1\} \to \{1, \ldots, n\}$ and denote by φ' its restriction to $\{1, \ldots, s\}$. By the induction hypothesis, we need to show that $T^{\Lambda}_{\varphi} = T^{\Lambda}_{\varphi'} \otimes_{\Lambda} C_{\varphi(s+1)}$.

Recall that the tensor product of two complexes X_{Λ} and ${}_{\Lambda}Y$ is defined by

$$(X \otimes_{\Lambda} Y)^m = \bigoplus_{p+q=m} X^p \otimes_{\Lambda} Y^q$$

with the differentials $d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$ for $x \in X^p$, $y \in Y^q$. It follows that for any $0 \le r \le s+1$, the term at degree $-r$ of $T^{\Lambda}_{\varphi'} \otimes_{\Lambda} C_{\varphi(s+1)}$ equals

$$T^{\Lambda,r}_{\varphi'} \oplus \Big(T^{\Lambda,r-1}_{\varphi'} \otimes_{\Lambda} \big(\Lambda e_{\varphi(s+1)} \otimes e_{\varphi(s+1)} \Lambda \big) \Big)$$

where the left summand vanishes for $r = s+1$ and the right vanishes for $r = 0$. Expanding these summands according to (2.1), we get a sum over all the r-tuples $1 \le i_1 < \dots < i_r \le s+1$, where the left summand corresponds to the tuples with $i_r \le s$ while the right to the tuples with $i_r = s+1$. Hence the term equals $T^{\Lambda,r}_{\varphi}$.

Concerning the differentials, we have the following picture

$$\begin{array}{ccccc} T^{\Lambda,r}_{\varphi} & = & T^{\Lambda,r}_{\varphi'} & \oplus & T^{\Lambda,r-1}_{\varphi'} \otimes_{\Lambda} \big(\Lambda e_{\varphi(s+1)} \otimes e_{\varphi(s+1)} \Lambda \big) \\ & & \Big\downarrow d^r_{\varphi'} & \swarrow (-1)^{r-1} \otimes m & \Big\downarrow d^{r-1}_{\varphi'} \otimes 1 \\ T^{\Lambda,r-1}_{\varphi} & = & T^{\Lambda,r-1}_{\varphi'} & \oplus & T^{\Lambda,r-2}_{\varphi'} \otimes_{\Lambda} \big(\Lambda e_{\varphi(s+1)} \otimes e_{\varphi(s+1)} \Lambda \big) \end{array}$$

which shows that they coincide with the d^r_φ as defined in (2.2). □

As a side application, we show the following commutativity result which is analogous to the fact that in a Weyl group corresponding to a generalized Cartan matrix B, the two simple reflections r_i^B and r_j^B commute when $B_{ij} = 0 = B_{ji}$, compare with Proposition 2.12 of [20].

Lemma 2.4. *If $\langle P_i, P_j \rangle_\Lambda = 0 = \langle P_j, P_i \rangle_\Lambda$ then $R_i^\Lambda R_j^\Lambda \simeq R_j^\Lambda R_i^\Lambda$.*

Proof. Indeed, $R_i^\Lambda R_j^\Lambda$ and $R_j^\Lambda R_i^\Lambda$ are given by the complexes

$$\Lambda e_j \otimes e_j \Lambda e_i \otimes e_i \Lambda \to (\Lambda e_j \otimes e_j \Lambda) \oplus (\Lambda e_i \otimes e_i \Lambda) \to \Lambda,$$
$$\Lambda e_i \otimes e_i \Lambda e_j \otimes e_j \Lambda \to (\Lambda e_i \otimes e_i \Lambda) \oplus (\Lambda e_j \otimes e_j \Lambda) \to \Lambda$$

which are isomorphic since $e_j \Lambda e_i = 0 = e_i \Lambda e_j$. □

A special role is played by the composition $R_n^\Lambda \cdot \ldots \cdot R_2^\Lambda \cdot R_1^\Lambda$ corresponding to the identity function on $\{1, \ldots, n\}$. We thus denote by $T^\Lambda = T^\Lambda_{id}$ the corresponding complex of bimodules of Lemma 2.3, so that

$$R_n^\Lambda \cdot \ldots \cdot R_2^\Lambda \cdot R_1^\Lambda = - \overset{\mathbf{L}}{\otimes}_\Lambda T^\Lambda. \tag{2.3}$$

2.2. **Triangular algebras.** In this section we study the complexes T^A for triangular algebras A.

Definition 2.5. A finite dimensional algebra A over a field k, with primitive orthogonal idempotents $e_1, \ldots, e_n$, is called *triangular* if $e_i A e_j = 0$ for all $j < i$ and $e_i A e_i \simeq k$ for all $1 \le i \le n$.

Triangular algebras have finite global dimension, hence the categories per A and $\mathcal{D}^b(A)$ coincide.

Lemma 2.6. *Let A be triangular and let $1 \le i \le j \le n$. Then*

$$R_i^A(P_j) \simeq \begin{cases} 0 & \text{if } j = i, \\ P_j & \text{if } j > i, \end{cases}$$

in the homotopy category $\mathcal{K}^b(A)$.

Proof. If $i < j$, then $\operatorname{Hom}_A(P_i, P_j) \simeq e_j A e_i = 0$, hence by Lemma 2.1, $R_i^A(P_j) = P_j$ (even in $\mathcal{C}^b(A)$).

Similarly, $\operatorname{Hom}_A(P_i, P_i) \simeq k$, hence $R_i^A(P_i)$ equals the null-homotopic complex $P_i \to P_i$, so it vanishes in $\mathcal{K}^b(A)$. □

Proposition 2.7. *Let A be triangular. Then:*

(a) *The functor $R_n^A \cdot \ldots \cdot R_2^A \cdot R_1^A$ on $\mathcal{D}^b(A)$ is isomorphic to the zero functor.*
(b) *$T^A \simeq 0$ in $\mathcal{D}^b(A^{op} \otimes A)$.*
(c) *T^A is contractible as a complex of right A-modules as well as a complex of left A-modules.*

Proof. A repeated application of Lemma 2.6 shows that for $1 \le j, s \le n$,

$$(R_s^A \cdot \ldots \cdot R_1^A)(P_j) \simeq \begin{cases} 0 & \text{if } j \le s, \\ P_j & \text{if } j > s, \end{cases}$$

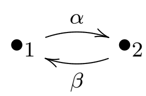
in $\mathcal{K}^b(A)$, hence the complex $(R_n^A \cdot \ldots \cdot R_1^A)(A)$ is homotopic to zero. Since A generates $\mathcal{D}^b(A)$, the first assertion follows. Now the second assertion follows from (2.3). For the third, observe that all the terms of T^A are projective both as right and as left A-modules (in fact, the above argument shows directly the contractibility of T^A as a complex of right A-modules). □

Remark 2.8. Since all its terms at negative degrees are also projective as A-A-bimodules, the complex T^A yields a projective resolution of A as an A-A-bimodule, which can be useful when computing Hochschild cohomology. Indeed, a similar resolution is given by Cibils [7], where an explicit contraction (of k-modules) is also given.

Remark 2.9. Since T^A is contractible as a complex of left A-modules, the tensor product $X \otimes_A T^A$ yields a projective resolution of a right module X_A. Similarly, $T^A \otimes_A Y$ gives a projective resolution of a left module ${}_AY$.

The statement of Proposition 2.7 is no longer true when the assumption that A is triangular is removed, even under the condition that A has finite global dimension. This is demonstrated by the following example.

Example 2.10. Let Λ be the path algebra of the quiver



modulo the ideal generated by the path $\beta\alpha$. The algebra Λ is 5-dimensional, its primitive orthogonal idempotents e_1, e_2 correspond to the vertices of the quiver and its global dimension is 2. However, Λ is not triangular as its Cartan matrix equals

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Moreover, the complex

$$T^\Lambda = \Big(\Lambda e_1 \otimes e_1 \Lambda e_2 \otimes e_2 \Lambda \to (\Lambda e_1 \otimes e_1 \Lambda) \oplus (\Lambda e_2 \otimes e_2 \Lambda) \to \Lambda\Big)$$

is not acyclic since its Euler characteristic as a complex of vector spaces (that is, the alternating sum of dimensions) is $3 \cdot 1 \cdot 2 - (3 \cdot 3 + 2 \cdot 2) + 5 \neq 0$.

Note that when $k = \mathbb{C}$, the category $\operatorname{mod} \Lambda$ is equivalent to the principal block of category $\mathcal{O}$ of the complex Lie algebra $\mathfrak{sl}_2$, see [21, §5.1.1].

For a triangular algebra A, the compositions of R_i^A in the *reverse* order have a very simple form. This is recorded in the next proposition, which will not be used in the sequel.

Proposition 2.11. *Let A be triangular. Let $I \subseteq \{1, \ldots, n\}$ and enumerate its elements in decreasing order $I = \{i_1 > i_2 > \cdots > i_s\}$. Then*

$$R_{i_s}^A \cdot \ldots \cdot R_{i_1}^A = - \overset{\mathbf{L}}{\otimes}_A \Big(\bigoplus_{i \in I} A e_i \otimes e_i A \xrightarrow{m} A \Big)$$

Proof. Apply Lemma 2.3 for the function φ defined by $\varphi(t) = i_t$ for $1 \leq t \leq s$ and observe that all the terms $T_\varphi^{A,r}$ vanish when $r > 1$ as $e_{i_t} A e_{i_{t+1}} = 0$ for all $1 \leq t < s$. □

2.3. **Triangular bimodules and their trivial extensions.** Let A be a basic finite-dimensional algebra with primitive orthogonal idempotents $e_1, \dots, e_n$.

Definition 2.12. An A-A-bimodule M is *triangular* if $e_i M e_j = 0$ for all $j < i$.

Let M be a triangular bimodule and let $DM = \operatorname{Hom}_k(M, k)$ be its dual. Consider the *trivial extension* $\Lambda = A \ltimes DM$, that is, the k-algebra which has $A \oplus DM$ as an underlying vector space, with the multiplication defined by $(a, \mu)(a', \mu') = (aa', a\mu' + \mu a')$.

The ring homomorphisms $A \xrightarrow{\iota} \Lambda \xrightarrow{\pi} A$ given by

$$\iota(a) = (a, 0) \qquad\qquad \pi(a, \mu) = a$$

give rise to the bimodules ${}_A\Lambda_A$ and ${}_\Lambda A_\Lambda$ (where $a \in A$ acts via multiplication by $\iota(a)$ and $\lambda \in \Lambda$ acts via multiplication by $\pi(\lambda)$). In particular we have the exact functors

$$\iota^* = - \otimes_\Lambda \Lambda_A = \operatorname{Hom}_\Lambda({}_A\Lambda_\Lambda, -) : \operatorname{mod} \Lambda \to \operatorname{mod} A$$
$$\pi^* = - \otimes_A A_\Lambda = \operatorname{Hom}_A({}_\Lambda A_A, -) : \operatorname{mod} A \to \operatorname{mod} \Lambda$$

which induce functors

$$\mathcal{D}^b(\Lambda) \xrightarrow{-\otimes_\Lambda \Lambda_A} \mathcal{D}^b(A), \qquad\qquad \mathcal{D}^b(A) \xrightarrow{-\otimes_A A_\Lambda} \mathcal{D}^b(\Lambda).$$

The left derived functors of their adjoints

$$- \otimes_A \Lambda_\Lambda : \operatorname{mod} A \to \operatorname{mod} \Lambda, \qquad\qquad - \otimes_\Lambda A_A : \operatorname{mod} \Lambda \to \operatorname{mod} A.$$

give rise to

$$\mathcal{D}^b(A) = \operatorname{per} A \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A \Lambda_\Lambda} \operatorname{per} \Lambda, \qquad \operatorname{per} \Lambda \xrightarrow{-\overset{\mathbf{L}}{\otimes}_\Lambda A_A} \operatorname{per} A = \mathcal{D}^b(A).$$

The elements $\iota(e_1), \dots, \iota(e_n)$ are primitive orthogonal idempotents of Λ. We shall denote them by $e_1, \dots, e_n$ when there is no risk of confusion.

Proposition 2.13. *Let A be a finite-dimensional basic algebra and M be a triangular bimodule. Then there exist short exact sequences of complexes of bi-modules*

$$\begin{aligned} &0 \to {}_\Lambda DM_A \to \Lambda \otimes_A T^A \to T^\Lambda \otimes_\Lambda A \to 0 \\ &0 \to {}_A DM_\Lambda \to T^A \otimes_A \Lambda \to A \otimes_\Lambda T^\Lambda \to 0 \end{aligned} \tag{2.4}$$

Proof. We prove only the exactness of the first sequence, as the proof for the other is similar.

Let $1 \leq r \leq n$ and consider the terms at degree $-r$ of $\Lambda \otimes_A T^A$ and $T^\Lambda \otimes_\Lambda A$ as direct sums

$$\left(\Lambda \otimes_A T^A\right)^{-r} = \bigoplus \Lambda e_{i_1} \otimes e_{i_1} A e_{i_2} \otimes \dots \otimes e_{i_{r-1}} A e_{i_r} \otimes e_{i_r} A$$
$$\left(T^\Lambda \otimes_\Lambda A\right)^{-r} = \bigoplus \Lambda e_{i_1} \otimes e_{i_1} \Lambda e_{i_2} \otimes \dots \otimes e_{i_{r-1}} \Lambda e_{i_r} \otimes e_{i_r} A$$

running over the tuples $1 \leq i_1 < \dots < i_r \leq n$.

By our hypothesis that M is a triangular bimodule, $e_j M e_i = 0$ hence $e_i DM e_j = 0$ for all $i < j$. Therefore we can identify $e_i A e_j$ with $e_i \Lambda e_j$ (via either ι or π) so that the terms $(\Lambda \otimes_A T^A)^{-r}$ and $(T^\Lambda \otimes_\Lambda A)^{-r}$ are isomorphic via the map

$$\lambda_0 \otimes a_1 \otimes \ldots \otimes a_{r-1} \otimes a_r \mapsto \lambda_0 \otimes \iota(a_1) \otimes \ldots \otimes \iota(a_{r-1}) \otimes a_r.$$

Moreover, by considering the explicit forms of the right A-action on Λ and the left Λ-action on A,

$$\lambda_0 \cdot a_1 = \lambda_0 \iota(a_1), \qquad a_{r-1} a_r = \iota(a_{r-1}) \cdot a_r, \qquad \iota(a_j a_{j+1}) = \iota(a_j)\iota(a_{j+1})$$

for $1 \leq j < r-1$, we see that these isomorphisms commute with the differentials as long as $r > 1$.

Finally, note that $(\Lambda \otimes_A T^A)^0 = \Lambda$, $(T^\Lambda \otimes_\Lambda A)^0 = A$ and there is a commutative diagram

$$\begin{array}{ccc} \bigoplus \Lambda e_i \otimes e_i A & \xrightarrow{d^{A,1}} & \Lambda \\ \downarrow \simeq & & \downarrow \pi \\ \bigoplus \Lambda e_i \otimes e_i A & \xrightarrow{d^{\Lambda,1}} & A \end{array}$$

with the top and bottom differentials given by

$$d^{A,1} : \lambda_i \otimes a_i \mapsto \lambda_i \iota(a_i) \in \Lambda, \qquad d^{\Lambda,1} : \lambda_i \otimes a_i \mapsto \pi(\lambda_i) a_i \in A$$

respectively.

Summarizing, we get the following commutative diagram of complexes of A-Λ-bimodules which shows the desired exact sequence.

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & DM \\ & & & & & & \downarrow \\ \cdots \to & \bigoplus \Lambda e_{i_1} \otimes e_{i_1} A e_{i_2} \otimes \ldots \otimes e_{i_{r-1}} A e_{i_r} \otimes e_{i_r} A & \to & \cdots \to & \bigoplus \Lambda e_i \otimes e_i A & \longrightarrow & \Lambda \\ & \downarrow \simeq & & & \downarrow \simeq & & \downarrow \pi \\ \cdots \to & \bigoplus \Lambda e_{i_1} \otimes e_{i_1} \Lambda e_{i_2} \otimes \ldots \otimes e_{i_{r-1}} \Lambda e_{i_r} \otimes e_{i_r} A & \to & \cdots \to & \bigoplus \Lambda e_i \otimes e_i A & \longrightarrow & A \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

□

2.4. **Proof of Theorem 1.2.** Let A be a triangular algebra with primitive orthogonal idempotents $e_1, \ldots, e_n$ and let M be a triangular A-A-bimodule (with respect to this ordering of the idempotents). In this case, we can combine Propositions 2.7 and 2.13 and deduce the following result.

Corollary 2.14. *Let $\Lambda = A \ltimes DM$. We have*

$$T^\Lambda \otimes_\Lambda A \xrightarrow{\sim} DM[1] \qquad A \otimes_\Lambda T^\Lambda \xrightarrow{\sim} DM[1]$$

in $\mathcal{D}^b(\Lambda^{op} \otimes A)$ and $\mathcal{D}^b(A^{op} \otimes \Lambda)$, respectively.

Proof. Since T^A is contractible as a complex of left A-modules, the complex $\Lambda \otimes_A T^A$ is contractible as a complex of left Λ-modules, hence it is isomorphic to zero in $\mathcal{D}^b(\Lambda^{op} \otimes A)$. Now the assertion follows from the first short exact sequence in (2.4). The proof of the second assertion is similar. □

Part (b) of Theorem 1.2 now follows from Corollary 2.14 by setting $R_i = R_i^{\Lambda}$ for $1 \leq i \leq n$ and using (2.3).

Remark 2.15. When M is zero, $\Lambda = A$ and we recover Proposition 2.7.

2.5. ***K*-theoretic interpretation.** We now prove part (a) of Theorem 1.2 and explain how that theorem can be regarded as a categorification of equation (1.1). In fact, we will recover this equation through a process known as *decategorification*, by looking at the effect of the functors appearing in the theorem on the corresponding Grothendieck groups.

Indeed, as the functors R_i^{Λ}, $- \overset{\mathbf{L}}{\otimes}_A DM_A[1]$ and $- \overset{\mathbf{L}}{\otimes}_\Lambda A$ are triangulated, they induce linear maps on the corresponding Grothendieck groups, which we describe explicitly in terms of the so-called Cartan matrices of A and Λ.

For an arbitrary (basic) finite dimensional algebra Λ with indecomposable projectives $P_1, \dots, P_n$, it will be convenient to reorder them in reverse order and to consider the basis

$$\varepsilon_1 = [P_n], \varepsilon_2 = [P_{n-1}], \dots, \varepsilon_n = [P_1]$$

of the Grothendieck group $K_0(\operatorname{per} \Lambda)$. We denote by C_Λ the matrix of the Euler form $\langle \cdot, \cdot \rangle_\Lambda$ with respect to that basis, known as the *Cartan matrix* of Λ. In explicit terms,

$$\begin{aligned}(C_\Lambda)_{ij} = \langle P_{n+1-i}, P_{n+1-j} \rangle_\Lambda &= \dim_k \operatorname{Hom}_\Lambda(P_{n+1-i}, P_{n+1-j}) \\ &= \dim_k e_{n+1-j} \Lambda e_{n+1-i}.\end{aligned}$$

Lemma 2.16. *Let $1 \leq i \leq n$. Then the matrix of the linear map on $K_0(\operatorname{per} \Lambda)$ induced by R_i^{Λ} is given by $r_{n+1-i}^{C_\Lambda}$.*

Proof. The j-th column of that matrix is equal to the class of $R_i^{\Lambda}(P_{n+1-j})$ in $K_0(\operatorname{per} \Lambda)$ which, according to Corollary 2.2, equals

$$[R_i^{\Lambda}(P_{n+1-j})] = [P_{n+1-j}] - \langle P_i, P_{n+1-j} \rangle_\Lambda [P_i] = \varepsilon_j - (C_\Lambda)_{n+1-i,j} \varepsilon_{n+1-i}.$$

□

For an algebra A with primitive orthogonal idempotents $e_1, \dots, e_n$, the condition that A is triangular implies that the matrix C_A is upper triangular with ones on its main diagonal. Similarly to the definition of C_A, one can define for any A-A-bimodule X, a Cartan matrix C_X by

$$(C_X)_{ij} = \dim_k e_{n+1-j} X e_{n+1-i},$$

so that X is triangular if and only if C_X is upper triangular.

Lemma 2.17. *Let A be a triangular algebra and X an A-A-bimodule. Then the matrix of the linear map on $K_0(\operatorname{per} A)$ induced by the functor $- \overset{\mathbf{L}}{\otimes}_A X$ is given by $C_A^{-1} C_X$.*

Proof. Denote that matrix (with respect to the basis $\varepsilon_1, \dots, \varepsilon_n$) by x. Since the functor $- \overset{\mathbf{L}}{\otimes}_A X$ sends each P_j to $P_j \otimes_A X \simeq e_j X$, we have

$$[e_{n+1-j} X] = \sum_{i=1}^{n} x_{ij} [P_{n+1-i}]$$

for all $1 \leq i \leq n$. Now, for any $1 \leq l \leq n$,

$$\begin{aligned}(C_X)_{lj} &= \dim_k e_{n+1-j} X e_{n+1-l} = \langle P_{n+1-l}, e_{n+1-j}X \rangle_A \\ &= \sum_{i=1}^{n} x_{ij} \langle P_{n+1-l}, P_{n+1-i} \rangle_A = \sum_{i=1}^{n} (C_A)_{li} x_{ij},\end{aligned}$$

hence $C_X = C_A x$. ☐

When A is triangular and M is a triangular bimodule, the Cartan matrix of the trivial extension $\Lambda = A \ltimes DM$ equals $C_\Lambda = C_A + C_{DM} = C_A + C_M^T$. Hence $(C_\Lambda)_+ = C_A$ is the upper triangular part of C_Λ and $(C_\Lambda)_- = C_M$ is its lower triangular part, as defined in Proposition 1.1.

Combining everything together, observing that the functor $- \overset{\mathbf{L}}{\otimes}_\Lambda A$ sends the projective $\iota(e_i)\Lambda$ to $e_i A$ and thus induces the identity matrix between the isomorphic groups $K_0(\operatorname{per} \Lambda)$ and $K_0(\operatorname{per} A)$, we conclude the following.

Corollary 2.18. *The left diagram of Theorem 1.2 induces a commutative diagram on the Grothendieck groups*

$$\begin{array}{ccccccc} K_0(\operatorname{per}\Lambda) & \xrightarrow{r_n^{C_\Lambda}} & K_0(\operatorname{per}\Lambda) & \xrightarrow{r_{n-1}^{C_\Lambda}} & \cdots & \xrightarrow{r_1^{C_\Lambda}} & K_0(\operatorname{per}\Lambda) \\ \big\downarrow{\scriptstyle I_n} & & & & & & \big\downarrow{\scriptstyle I_n} \\ K_0(A) & & & \xrightarrow{\quad -(C_\Lambda)_+^{-1}(C_\Lambda)_-^T \quad} & & & K_0(A) \end{array}$$

(where I_n is the $n \times n$ identity matrix), thus recovering equation (1.1).

2.6. **Proof of Corollary 1.3.** Let $e_1, \dots, e_n$ be the primitive orthogonal idempotents of A and set $R_i = R_i^{T(A)}$ for $1 \leq i \leq n$.

The algebra $T(A)$ is symmetric and $\dim_k e_i T(A) e_i = 2$ for any $1 \leq i \leq n$. Hence by [13, Remark 4.3], the functors $R_i^{T(A)}$ are autoequivalences, see also [19, Theorem 4.1].

Since $\nu_A = - \overset{\mathbf{L}}{\otimes}_A DA$ and $A \ltimes DA = T(A)$, Corollary 1.3 is just a special case of Theorem 1.2, where the triangular bimodule M is taken to be A.

Remark 2.19. The Cartan matrix B of $T(A)$ is symmetric with 2 on its main diagonal, hence the matrices r_i^B are reflections. As the action of each autoequivalence $R_i^{T(A)}$ on $K_0(\operatorname{per} T(A))$ is given by a reflection, one may interpret this corollary as lifting of the Serre functor (up to a shift by one) on $\mathcal{D}^b(A)$ to a product of "reflection" functors on $\operatorname{per} T(A)$.

2.7. **Realization of matrices as Cartan matrices.** We now show that Theorem 1.2 categorifies (1.1) for any integral square matrix B with non-negative entries and positive entries on its main diagonal. We start with an observation on the realization of such matrices as Cartan matrices of finite dimensional algebras.

Lemma 2.20. *Let C be an integral $n \times n$ matrix with $C_{ij} \geq 0$ and $C_{ii} > 0$ for $1 \leq i, j \leq n$, and let k be a field. Then there exists a finite dimensional algebra A over k whose Cartan matrix equals C.*

Proof. We construct A from a quiver with relations. Let Q be the quiver whose vertices are $\{1, 2, \dots, n\}$, with the number of arrows from i to j set to

$$|\{\text{arrows } i \to j\}| = C_{n+1-j,n+1-i} - \delta_{ij}.$$

Let $I \subseteq kQ$ be the ideal in the path algebra of Q generated by all the paths of length 2. Then the Cartan matrix of $A = kQ/I$ equals C, since $\dim_k \operatorname{Hom}_A(P_{n+1-i}, P_{n+1-j})$ is the number of paths in Q from $n+1-j$ to $n+1-i$ of length at most one, which equals C_{ij} by construction. □

Observe that the algebra A constructed in the lemma is triangular if and only if C is upper triangular with ones on its main diagonal. We now consider the realization of bimodules with prescribed Cartan matrix over such algebras.

Lemma 2.21. *Let C' be an integral $n \times n$ matrix with non-negative entries and A a finite dimensional algebra as constructed in the previous lemma, which is furthermore triangular. Then there exists a bimodule M over A such that $C_M = C'$.*

Proof. Let $A = kQ/I$ be as in the previous lemma, and consider the quiver $Q^{op} \times Q$ whose vertices are the pairs (i, j) for $1 \leq i, j \leq n$, with "horizontal" arrows $(i, j) \to (i, j')$ for every arrow $j \to j'$ in Q and "vertical" arrows $(i', j) \to (i, j)$ for every arrow $i \to i'$ in Q. By our assumption on A, there are no loops in this quiver.

We construct a representation M of $Q^{op} \times Q$ as follows. For any vertex (i, j), let $M_{(i,j)}$ be a k-vector space of dimension $C'_{n+1-j,n+1-i}$. For any arrow $\alpha : (i, j) \to (i', j')$, take the linear map $M_\alpha : M_{(i,j)} \to M_{(i',j')}$ to be zero if $i \neq j$ and arbitrary otherwise.

Observe that for any path in $Q^{op} \times Q$ of length 2, the corresponding map in M vanishes, hence in particular M can be viewed as a bimodule over A. Moreover, $C_M = C'$ since

$$(C_M)_{ij} = \dim_k e_{n+1-j} M e_{n+1-i} = \dim_k M_{(n+1-j,n+1-i)} = C'_{ij}$$

by construction. □

Note that in the proof we could have taken all the maps M_α to be zero. However, the construction presented in the lemma is slightly more general and in particular one can realize A as a bimodule over itself in this way.

Combining the above two observations, we deduce the following.

Corollary 2.22. *Let B be an integral $n \times n$ matrix with $B_{ij} \geq 0$ and $B_{ii} > 0$ for $1 \leq i, j \leq n$, and let k be a field. Then there exist a finite dimensional triangular algebra A over k and a triangular bimodule M over A such that $C_A = B_+$ and $C_M = B_-$. In particular, $B = C_\Lambda$ for $\Lambda = A \ltimes DM$.*

Proof. Use Lemma 2.20 with $C = B_+$ to construct the algebra A, and then Lemma 2.21 with $C' = B_-$ to construct the bimodule M. □

In particular we see that Corollary 1.3 categorifies (1.3) for any square upper triangular integer matrix C with non-negative entries and 1 on its main diagonal (or equivalently, for any square symmetric integer matrix B with non-negative entries and 2 on its main diagonal).

3. Discussion and Comparison

In this section we recall previous work on path algebras of quivers that can be considered as a categorical interpretation of equation (1.3), and compare it with our approach.

3.1. **A result of Gabriel.** Fix an algebraically closed field k. For a quiver Q without oriented cycles, denote by kQ its path algebra and by $\mathcal{D}^b(Q)$ the bounded derived category of finite-dimensional right kQ-modules. Recall that a vertex $s \in Q$ is called a *sink* if there are no arrows in Q starting at s. The *reflection* of Q at s, denoted $\sigma_s Q$, is the quiver obtained from Q by inverting all the arrows ending at s while leaving all the others intact, so that s becomes a *source* in $\sigma_s Q$.

In [3], Bernstein, Gelfand and Ponomarev defined the *reflection functor* from the category of representations of Q to those of $\sigma_s Q$ (where s is a sink in Q). In the language of derived categories (see for example [11, (IV.4, Exercise 6)]), this functor induces a derived equivalence

$$R_s : \mathcal{D}^b(Q) \xrightarrow{\sim} \mathcal{D}^b(\sigma_s Q).$$

Order now the vertices of Q in an *admissible ordering*, that is, enumerate them in a sequence $1, 2, \dots, n$ such that there are no arrows $j \to i$ in Q for $i < j$. In this case, the vertex $1 \leq i \leq n$ is a sink in the quiver $\sigma_{i+1}\sigma_{i+2} \dots \sigma_n Q$. Moreover, the quiver $\sigma_1 \dots \sigma_n Q$ is equal to Q. Thus, the composition of the (derived) reflection functors

$$\mathcal{D}^b(Q) \xrightarrow{R_n} \mathcal{D}^b(\sigma_n Q) \xrightarrow{R_{n-1}} \mathcal{D}^b(\sigma_{n-1}\sigma_n Q) \xrightarrow{R_{n-2}} \dots \xrightarrow{R_1} \mathcal{D}^b(\sigma_1 \dots \sigma_n Q)$$

defines an autoequivalence $R_1 \cdot R_2 \cdot \dots \cdot R_n$ of $\mathcal{D}^b(Q)$, known as the *Coxeter functor*.

Another autoequivalence of $\mathcal{D}^b(Q)$ is given by the *Auslander-Reiten translation* τ, which is related to the Serre functor $\nu = - \overset{\mathbf{L}}{\otimes}_{kQ} D(kQ)$ on $\mathcal{D}^b(Q)$ by $\tau = \nu[-1]$, see [12]. The following result of Gabriel [10] relates it with the Coxeter functor.

Theorem 3.1 ([10])**.** *If the underlying graph of Q is a tree, or more generally, does not contain a cycle of odd length, then*

$$\tau \simeq R_1 \cdot R_2 \cdot \dots \cdot R_n. \tag{3.1}$$

Similarly to Corollary 2.18, the relation with equation (1.3) is obtained through decategorification, by considering the Grothendieck group $K_0(Q)$ of the triangulated category $\mathcal{D}^b(Q)$ together with its Euler form $\langle \cdot, \cdot \rangle_{kQ}$, but this time using bases of *simple* modules rather than the indecomposable projective ones.

Let S_i be the simple module corresponding to the vertex $1 \leq i \leq n$. The classes $[S_1], \dots, [S_n]$ form a basis of $K_0(Q)$, and the matrix C_Q of $\langle \cdot, \cdot \rangle_{kQ}$ with respect to that basis has an explicit combinatorial description, namely

$$(C_Q)_{ij} = \delta_{ij} - \big|\{\text{arrows } i \to j\}\big|.$$

When the vertices are ordered in an admissible order, the matrix C_Q is upper triangular with ones on its main diagonal.

It is well known that the matrix of the linear map on $K_0(Q)$ induced by τ is given by $-C_Q^{-1}C_Q^T$. On the other hand, for a sink s, the reflection functor R_s induces a linear map $K_0(Q) \to K_0(\sigma_s Q)$ whose matrix with respect to the bases of simples is given by the reflection $r_s^{B_Q}$, where $B_Q = C_Q + C_Q^T$ is the symmetrization of C_Q, see [3]. Moreover, $B_{\sigma_s Q} = B_Q$ since

$$(B_Q)_{ij} = 2\delta_{ij} - \left|\{\text{arrows } i \to j\}\right| - \left|\{\text{arrows } j \to i\}\right|$$

is independent on the orientation of the arrows.

Therefore, by looking at the Grothendieck groups, Theorem 3.1 implies the following commutative diagram

$$\begin{array}{ccccccc}
K_0(Q) & \xrightarrow{r_n^{B_Q}} & K_0(\sigma_n Q) & \xrightarrow{r_{n-1}^{B_Q}} & \cdots & \xrightarrow{r_1^{B_Q}} & K_0(\sigma_1 \dots \sigma_n Q) \\
\downarrow{\scriptstyle I_n} & & & & & & \downarrow{\scriptstyle I_n} \\
K_0(Q) & & & \xrightarrow{-C_Q^{-1}C_Q^T} & & & K_0(Q),
\end{array}$$

recovering equation (1.3) for $C = C_Q$ as a K-theoretical consequence of the isomorphism of the functors τ and $R_1 \cdot R_2 \cdot \ldots \cdot R_n$ on $\mathcal{D}^b(Q)$.

3.2. **Comparison.** While Theorem 1.2 and its Corollary 1.3, on the one hand, and Theorem 3.1, on the other hand, can all be regarded as categorical interpretations of equations (1.1) and (1.3), there are several differences, which are outlined below.

3.2.1. *Scope.* Compared with Theorem 3.1, Theorem 1.2 has broader scope in two aspects; first, it applies to any finite dimensional triangular algebra A, and not only to hereditary ones. Second, it applies to any triangular bimodule M, and not only to $M = A$, thus providing an interpretation of equation (1.1) rather than the more specific (1.3).

3.2.2. *Lifting vs. factorization.* This broader scope carries some price to be paid, namely that while Theorem 3.1 provides a factorization of the Auslander-Reiten translation as a composition of the reflection functors, Theorem 1.2 does not factor $- \overset{\mathbf{L}}{\otimes}_A DM[1]$, but rather provides only a factorization of a lift to $\operatorname{per} \Lambda$ for $\Lambda = A \ltimes DM$.

3.2.3. *The choice of matrix C.* Both Corollary 1.3 and Theorem 3.1 categorify the same statement, namely equation (1.3), and in both cases the upper triangular matrix C is the matrix of the Euler form with respect to some basis. However, in Corollary 1.3 this is the basis of indecomposable projectives, while in Theorem 3.1 it is the basis of simples.

The use of the basis of simples is a rather special feature of hereditary algebras. Indeed, recall that for a quiver Q and a sink s, one has $C_{\sigma_s Q} = r_s^T C_Q r_s$ where r_s is the corresponding reflection built from the symmetrization of C_Q. However, for a general triangular algebra A whose Euler form is given by an upper triangular matrix C with respect to the basis of simples, there may be no algebra whose Euler form is given by the matrix $r_s^T C r_s$, where s is a sink or a source in the quiver of A and r_s is the corresponding reflection built from the symmetrization of C.

Example 3.2. Let A be the algebra given by the quiver

$$\bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3$$

modulo the relation $\alpha\beta = 0$. The matrix of its Euler form with respect to the basis of simples $\{S_1, S_2, S_3\}$ is

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$r_1 = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

are the reflections built from the symmetrization $B = C + C^T$.

The matrices $r_1^T C r_1$ and $r_3^T C r_3$ cannot represent Euler forms of algebras with respect to bases of simples. Indeed, if this were the case then their inverses would be Cartan matrices of algebras, which is impossible since

$$\left(r_1^T C r_1\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \left(r_3^T C r_3\right)^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

contain negative entries.

3.2.4. *The shift.* Finally, observe that in both results, the minus sign in the left hand side of (1.1) and (1.3) is interpreted as a shift applied to the functor of tensoring with a bimodule. However, in Theorem 1.2 it is a positive shift, while in Theorem 3.1 it is a negative one. Of course, they are indistinguishable in the Grothendieck group.

References

1. Norbert A'Campo, *Sur les valeurs propres de la transformation de Coxeter*, Invent. Math. **33** (1976), no. 1, 61–67.
2. John C. Baez and James Dolan, *Categorification*, Higher category theory (Evanston, IL, 1997), Contemp. Math., vol. 230, Amer. Math. Soc., Providence, RI, 1998, pp. 1–36.
3. I.N. Bernstein, I.M. Gel'fand, and V.A. Ponomarev, *Coxeter functors and Gabriel's theorem.*, Russ. Math. Surv. **28** (1973), no. 2, 17–32 (English).
4. A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.
5. Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.
6. Sheila Brenner and M. C. R. Butler, *The equivalence of certain functors occurring in the representation theory of Artin algebras and species*, J. London Math. Soc. (2) **14** (1976), no. 1, 183–187.
7. Claude Cibils, *Cohomology of incidence algebras and simplicial complexes*, J. Pure Appl. Algebra **56** (1989), no. 3, 221–232.
8. A. J. Coleman, *Killing and the Coxeter transformation of Kac-Moody algebras*, Invent. Math. **95** (1989), no. 3, 447–477.
9. H. S. M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Math. J. **18** (1951), 765–782.

10. Peter Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 831, Springer, Berlin, 1980, pp. 1–71.
11. Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
12. Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
13. Mitsuo Hoshino and Yoshiaki Kato, *Tilting complexes defined by idempotents*, Comm. Algebra **30** (2002), no. 1, 83–100.
14. Robert B. Howlett, *Coxeter groups and M-matrices*, Bull. London Math. Soc. **14** (1982), no. 2, 137–141.
15. Mikhail Khovanov, Volodymyr Mazorchuk, and Catharina Stroppel, *A categorification of integral Specht modules*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1163–1169.
16. Sefi Ladkani, *On the periodicity of Coxeter transformations and the non-negativity of their Euler forms*, Linear Algebra Appl. **428** (2008), no. 4, 742–753.
17. Helmut Lenzing, *Coxeter transformations associated with finite-dimensional algebras*, Computational methods for representations of groups and algebras (Essen, 1997), Progr. Math., vol. 173, Birkhäuser, Basel, 1999, pp. 287–308.
18. Volodymyr Mazorchuk and Catharina Stroppel, *Projective-injective modules, Serre functors and symmetric algebras*, J. Reine Angew. Math. **616** (2008), 131–165.
19. Raphaël Rouquier and Alexander Zimmermann, *Picard groups for derived module categories*, Proc. London Math. Soc. (3) **87** (2003), no. 1, 197–225.
20. Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.
21. Catharina Stroppel, *Category $\mathcal{O}$: quivers and endomorphism rings of projectives*, Represent. Theory **7** (2003), 322–345 (electronic).

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: sefil@mpim-bonn.mpg.de