Theorem 10.1. Let $L \supset F \supset K$ be a finite field extension,$a \in L$. Then

$$a)Tr_{L/K}(a) = Tr_{F/K}(Tr_{L/F}(a))$$

and

$$b)N_{L/K}(a) = N_{F/K}(N_{L/F}a))$$

Proof. Let $\alpha_i \in L, 1 \leq i \leq n := [L : F]$ be a basis of L as an F-vector space and $\beta_j \in F, 1 \leq j \leq m := [F : K]$ be a basis of F as a K-vector space and

$l_{ij} := \{\alpha_i \beta_j\} \subset L, 1 \leq i \leq n, 1 \leq j \leq m$.

As we have seen in the first lecture the set $\{l_{ij}\} \subset L$ is a basis of L as an K-vector space.

For any $b \in F$ we denote by $M^b = (m^b_{ii'}), m^b_{ii'} \in K, 1 \leq j, j' \leq m$ the $m \times m$ matrix of the operator of the multiplication by b in F computed in the basis $\beta_j, 1 \leq j \leq m$. Analogously for any $a \in L$ we denote by $B^a = (b^a_{ii'}), b^a_{ii'} \in F, 1 \leq i, i' \leq n$ the $n \times n$ matrix of the operator of the multiplication by a in the F-vector space L computed in the basis $\alpha_i, 1 \leq i \leq n$.

Now for any $a \in L$ consider $n \times n$-matrix $C_a = (c_{ii'}), 1 \leq i, i' \leq n$ whose entries are $m \times m$-matrices $c_{ii'} := M^{b_{ii'}}$. We can naturally consider C_a as an $mn \times mn$-matrix $\tilde{C}_a$ with entries in K .

Lemma 10.1. The matrix $\tilde{C}_a$ is equal to the matrix of the operator of the multiplication by a in L computed in the basis $l_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$.

I'll leave the proof of the Lemma as an exercise.

Now it is easy to prove the part a) of the Theorem. By the definition

$$Tr_{L/K}(a) = Tr(\tilde{C}_a) = \sum_{1 \leq i \leq n} Tr(c_{ii})$$

$$= \sum_{1 \leq i \leq n} Tr_{F/K}((b_{ii})) = Tr_{F/K}(\sum_{1 \leq i \leq n} (b_{ii})) = Tr_{F/K}(Tr_{L/F}(a))$$

To prove the part b) we have to show that $Det(\tilde{C}_a) = N_{F/K}(Det(C_a))$ for any $a \in L$. As often happens it is easier to prove a more general result.

For any $n \times n$ matrix $B = (b_{ii'}), b_{ii'} \in F, 1 \leq i, i' \leq n$ we denote by $\tilde{B}$ the $n \times n$-matrix $\tilde{B} = (\tilde{b}_{ii'}), 1 \leq i, i' \leq n$ whose entries are $m \times m$-matricies $M^{b_{ii'}}$. We can naturally consider $\tilde{B}$ as an $mn \times mn$-matrix with entries in K.

lemma 10.2. The map $\phi : GL_m(F) \to GL_{mn}(K), B \to \tilde{B}$ is a group homomorphism.

I'll leave the proof of Lemma 10.2 as a homework.

Proposition 10.1..For any $n \times n$ matrix $B \in GL_n(F)$ we have $Det(\tilde{B}) = N_{F/K}(Det(B))$.

We start the proof with the special case. .

lemma 10.3. Show that the Proposition 10.1 is true
a) if B is an upper [or lower]triangular matrix,
b) if B is a permutation matrix..

I'll leave the proof of Lemma 10.3 as a homework.

Proof of the Proposition 10.1 . Consider two maps $f_1, f_2 : GL_n(F) \to K^*$ where $f_1(B) := N_{L/K}(Det(B)), f_2(A) := Det(\tilde{B})$. We want to show that $f_1 \equiv f_2$. Since the $\phi : GL_m(F) \to GL_{mn}(K), B \to \tilde{B}$ and the determinant maps are group homomorphism we see that both f_1 and f_2 are group homomorphism. As you know any invertible $n \times n$-matrix can be brought by the row reduction procedure to the reduced echelon form [this is known as the Gauss reduction procedure]. In other words any invertible $n \times n$-matrix B can be written in the form $B = A^+ s A^-$ where A^+ is an upper triangular matrix, A^- is an lower triangular matrix and s is a permutation matrix. Since both f_1 and f_2 are group homomorphism to prove the equality $f_1(B) = f_2(B)$ it is sufficient to check that $f_1(A^+) = f_2(A^+), f_1(A^-) = f_2(A^-)$ and $f_1(s) = f_2(s)$ where A^+ is an upper triangular matrix, A^+ is an lower triangular matrix and s is a permutation matrix. But this is an easy exercise.$\square$.

Now we can finish the proof of Theorem 10.1. By the definition $N_{N/L}(\alpha) = Det(M')$ and therefore $N_{L/K}(N_{N/L}(\alpha)) = N_{L/K}(Det(M))$. On the other hand $N_{N/K}(\alpha) = Det(M'')$ and the equality $N_{N/K}(\alpha) = N_{L/K}(N_{N/L}(\alpha))$ follows from Proposition 10.1.$\square$

Theorem 10.2. a) Let $L \supset K$ be a finite extension $\alpha \in L, \sigma_1, ..., \sigma_n$ the set of K-homomorphisms from L to $\bar{K}$. Then

a)

$$Tr_{L/K}(\alpha) = \sum_{1 \leq i \leq n} \sigma_i(\alpha)$$

if the extension $L \supset K$ is separable and
$Tr_{L/K}(\alpha) = 0$ otherwise,

b) $N_{L/K}(\alpha) = \prod_{1 \leq i \leq n} \sigma_i(\alpha)^{[L:K]_i}$.

Proof. I'll give a proof of Theorem 10.2 only for the case when the extension $L \supset K$ is separable. Assume first that $L = K(\alpha)$. Let $p(t) = Irr(\alpha, K, t)$. Since the extension $L \supset K$ is separable the polynomial $p(t)$ has $[L : K]$ distinct roots in $\bar{K}$ and it follows from Lemma 3.3 that these roots have a form $\alpha_i = \sigma_i(\alpha), 1 \leq i \leq n$. So in this case Theorem 10.1 follows from Lemma 9.1.

Consider now the case of an arbitrary separable extension $L \supset K$. I'll give only a proof of the part b). The part a) is only easier.

As follows from Theorem 10.1 we have $N_{L/K}(\alpha) = N_{K(\alpha)/K}^{[L:K(\alpha)]}(\alpha)$.

On the other hand let $\tau_1, ..., \tau_r, r = [K(\alpha) : K]$ be the set of K-homomorphism from $K(\alpha)$ to $\bar{K}$. As follows from the arguments used in the proof of Lemma 6.5 for any $j, 1 \leq j \leq r$ there exists $[L : K(\alpha)]$ K-homomorphism σ from L to $\bar{K}$ such that $\sigma(\beta) = \tau_j(\beta), \forall \beta \in K(\alpha)$. Therefore $\prod_{1 \leq i \leq n} \sigma_i(\alpha) = (\prod_{1 \leq j \leq r} \tau_j(\alpha))^{[L:K(\alpha)]}$. Since we already know that

$$\prod_{1 \leq j \leq r} \tau_j(\alpha) = N_{K(\alpha)/K}(\alpha)$$

we see that $N_{L/K}(\alpha) = \prod_{1 \leq i \leq n} \sigma_i(\alpha)$.□

Let $L \supset K$ be a finite extension. Consider L as a K-vector space and define a bilinear form

$$<,>: L \times L \to K$$

by $< \alpha, \beta > := Tr_{L/K}(\alpha\beta)$

Lemma 10.4. If the extension $L \supset K$ is separable then the bilinear form is non-degenerate.

Proof. We first show that the map $Tr_{L/K} : L \to K$ is not identically zero. Really as follows from the theorem 10.2 we have

$$Tr_{L/K}(\alpha) = \sum_{1 \leq i \leq n} \sigma_i(\alpha)$$

By Lemma of Dedekind we know that the maps $\sigma_i : L \to \bar{K}, 1 \leq i \leq n$ are linearly independent. Therefore we know that there exists $\alpha \in L$ such that $\sum_{1 \leq i \leq n} \sigma_i(\alpha) \neq 0$. In other words there exists $\gamma \in L$ such that $Tr_{L/K}(\gamma) \neq 0$.

By the definition a bilinear form $<,>: L \times L \to K$ on a finite-dimensional K-vector space is non-degenerate if for any $\alpha \in L - \{0\}$ there exists $\beta \in L$ such that $< \alpha, \beta > \neq 0$. But we can take $\beta = \gamma/\alpha$.□