HOMEWORK #1 IN ALGEBRAIC STRUCTURES 2

Problem 1. Show that

a) if $p(t), q(t) \in K[t]$ are two polynomials such that the corresponding principal ideals $(p(t))$ and $(q(t))$ coincide then there exists $c \in K - \{0\}$ such that $p(t) = cq(t)$,

b) for any two non-zero polynomials $p(t), q(t) \in K[t]$ we have deg $(p(t)q(t))$=deg $p(t)$+deg $q(t)$,

c) if $p(t)$ is a non-zero polynomial of degree n then it has no more then n distinct roots,

d) Show that given $p(t), q(t) \in K[t]$ such that $p(t) \neq 0$ there exists unique pair $a(t), r(t) \in K[t]$ such that
$q(t) = a(t)p(t) + r(t)$ and deg $r(t) <$ deg $p(t)$.

Remark The polynomial $r(t)$ is called the *remainder* of $q(t)$ after the division by $p(t)$.

Problem 2. a) Find $[\mathbb{C} : \mathbb{R}]$,

b) show that the extension $\mathbb{Q} \subset \mathbb{R}$ is not elementary,

c) Let L be a field and $K \subset L$ a subfield of L. Show that for any $\alpha \in L$ the set $K(\alpha) \subset L$ is a subfield of L,

d) Let L be a field and $K \subset L$ a subfield of L. Show that for any subset $A \subset L$ the set $K(A) \subset L$ is a subfield of L,

e) if $p(t) \in \mathbb{R}[t]$ is irreducible then either deg $p(t) = 1$ or deg $p(t) = 2$.

Remark. I assume that you know that any polynomial $p(t) \in \mathbb{C}[t]$ of positive degree has a root $a \in \mathbb{C}$.

Problem 3. Prove the part b) of the Theorem 1.1: Let L be a finite extension of F, F be a finite extension of K. Let $\alpha_i \in L, 1 \leq i \leq [L : F]$ be a basis of L as an F-vector space and let $\beta_j \in F, 1 \leq j \leq [F : K]$ be a basis of F as a K-vector space.

Prove that the elements $l_{ij} = \alpha_i\beta_j \in L$ for $1 \leq i \leq [L : F], 1 \leq j \leq [F : K]$ in the K-vector space L are linearly independent (over K).

Problem 4. Let $u \in \mathbb{C}$ be a solution of the equation

$$(\star)u^3 - u^2 + u + 2 = 0$$

and $E = \mathbb{Q}(u)$

a) show that $[E : \mathbb{Q}]$ does not depend on a choice u of a solution of $(\star)$,

b) express $(u^2 + u + 1)(u^2 - u)$ and $(u - 1)^{-1}$ in the form

$$au^2 + bu + c$$

where $a, b, c \in \mathbb{Q}$

Let $\xi_n \in \mathbb{C}$ be a primitive n-th root of 1. [That is $\xi_n^n = 1$ but $\xi_n^m \neq 1$ for all $1 \leq m < n$].

c) show that the subfield $L_n := \mathbb{Q}(\xi_n) \subset \mathbb{C}$ does not depend on a choice of a primitive root $\xi_n \in \mathbb{C}$,

d) find $[L_n : \mathbb{Q}]$ for $2 \leq n \leq 4$

Problem 5. Let L be an extension of $K, \alpha \in L$ an element algebraic over K. show that $[K(\alpha) : K]$ is the minimal degree of a non-zero polynomial $p(t) \in K[t]$ such that $p(\alpha) = 0$ and that the polynomial is irreducible.