Fundamental Notions in Algebra – Exercise. 12

1. Describe conjugacy classes, irreducible representations and write character tables of the following groups:

   (a) S_4 (the symmetric group on 4 letters)

   (b) A_4 (the alternating group on 4 letters)

   (c) $Q = \{1, -1, i, -i, j, -j, k, -k\}$ (the multiplicative subgroup of the quaternions $\mathbb{H}$).

2. Let R be a a division algebra such that $\forall r \in R \quad \exists n > 1$ such that $r^n = r$. Show that R is a field.
   Hint:

   (a) Show that R is an algebra over a finite field $\mathbb{F}_p$ for some prime p and that $\mathbb{F}_p[r]$ is a finite field for each $r \in R$.

   (b) Set $L = Z(R)$ and fix $r \in R$. Show that $L[r]$ is a finite Galois extension of L, and fix $\sigma \in Gal(L[r]/L)$. Show that there exists $k \in \mathbb{N}$ such that $\sigma(r) = r^{p^k}$

   (c) Show that there exists $s \in R$ such that $srs^{-1} = r^{p^k}$ and that r and s generate a finite division algebra.

   (d) Conclude that $L[r] = L$, thus $R = L$ is commutative.

3. Let $\rho_1 : G_1 \to \mathrm{Aut}_k(V_1)$ and $\rho_2 : G_2 \to \mathrm{Aut}_k(V_2)$ be two finite-dimensional representations.

   (a) Show that the character of $\rho := \rho_2 \boxtimes \rho_2$ satisfies that $\forall g_1 \in G_1\ ,\ g_2 \in G_2 \quad \chi_\rho(g_1, g_2) = \chi_{\rho_1}(g_1) \cdot \chi_{\rho_2}(g_2)$.

   (b) Let ρ be a representation $G_1 \times G_2 \to \mathrm{Aut}_k(\mathrm{Hom}_k(V_1, V_2))$ defined by the rule $\rho(g_1, g_2)(f) := \rho_2(g_2) \circ f \circ \rho_1(g_1)^{-1}$. Calculate χ_ρ in terms of ρ_1 and ρ_2.

4. Let ρ and ρ' be two irreducible finite dimensional representations of a group G $\big($not necessarily finite$\big)$ over an algebraically closed field k such that $\chi_\rho = \chi_{\rho'}$. Show that $\rho \sim \rho'$.

   Hint: Consider the represntation (τ, V) of G, where V is the k-vector space all functions $f : G \to k$ and $\tau(g)(f)(g') = f(g'g)$ for all $g, g' \in G$. Denote by τ_ρ the subrepresentation of τ generated by $\chi_\rho \in V$. Show that the space of τ_ρ is $Span\{\rho_{i,j}\}$ and conclude that τ_ρ is isomorphic to the direct sum of $deg(\rho)$ copies of ρ.

5. Let ρ be an n-dimensional irreducible representation over $\mathbb{C}$ of a finite group G.

   (a) Show that $\forall g \in G \quad \chi_\rho(g^{-1}) = \chi_\rho(g)^-$ $\big($complex conjugate$\big)$.

   (b) Show that $\forall z \in Z(G) \quad |\chi_\rho(z)| = n$.

   (c) Show that $\sum_{g \in G} |\chi_\rho(g)|^2 = |G|$

   (d) Show that $n^2 \leq |G/Z(G)|$.