Fundamental Notions in Algebra – Exercise No. 4

1. Let V be a vector space over a field F of countable dimension, let $R = \mathrm{End}_F(V)$ be the set of linear endomorphisms of V, and let $I = \mathrm{End}_F^f(V)$ the the set of linear endomorphisms of V of finite rank.
   (a) Show that I is a two-sided ideal of R, and the quotient ring R/I is simple.
   (b) Show that the ring R/I is not semisimple.

2. (a) Let M be an artinian module. Show that every submodule and factor module of M is artinian.
   (b) Let M be a module, $P_1 \supseteq P_2$ and N submodules, and assume that $P_1 \cap N = P_2 \cap N$ and $P_1 + N = P_2 + N$. Show that $P_1 = P_2$.
   (c) Show that if $N \subset M$ and M/N are artinian, then M is artinian.
   (d) Assume that R is an artinian ring. Show that every finitely generated R-module is artinian.
   (e) Which of the assertions (a),(c),(d) will remain true if one replaces the word "artinian" by "noetherian"?
   (f) Show that a module M has finite length if and only if it is both noetherian and artinian.

3. (a) Show that $\mathbb{Z}$ is a noetherian ring which is not artinian.
   (b) Let p be a prime number and let $\mathbb{Z}_{p^\infty}$ be the submodule of $\mathbb{Q}/\mathbb{Z}$ consisting of elements which are annihilated by some power of p. Show that $\mathbb{Z}_{p^\infty}$ is an artinian $\mathbb{Z}$-module which is not noetherian.
   (c) Give an example of a ring which is not noetherian.

4. Let M be an R-module.
   (a) Assume either that R is an artinian ring or that M is an artinian module. Show that M contains a simple submodule.
   (b) Is the assertion in (a) still true if one replaces the word "artinian" by "noetherian"?

5. Let K/k be an infinite field extension. Denote by R the subset of $\mathrm{Mat}_2(K)$ consisting of elements of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

   with $a, b \in K, c \in k$. Show that:
   (a) R is a subring of $\mathrm{Mat}_2(K)$;
   (b) The ring R is noetherian and artinian;
   (c) The ring R^{op} is neither noetherian nor artinian;
   (d) The rings R and R^{op} are not isomorphic.

6. (a) Prove Fitting's lemma: If M is an artinian module, and $f : M \to M$ is an injective homomorphism, then f is surjective.
   (b) Prove the dual statement: If M is a noetherian module, and $f : M \to M$ is a surjective homomorphism, then f is injective.