Fundamental Notions in Algebra – Exercise No. 9

1. Let L/K be a finite Galois extension, and $\Gamma = Gal(L/K)$. Show that $B^2(\Gamma, L^\times)$ is a subgroup of $Z^2(\Gamma, L^\times)$.

2. Let L/K be a Galois extension, and $\Gamma = Gal(L/K)$. For a simple central finite-dimensional algebra A over K such that $L \subset A$ and $C_A(L) = L$, we denote by $E = E(A)$ be normalizer $N_{A^\times}(L^\times) = \{a \in A^\times : aL^\times a^{-1} = L^\times\}$.

   (a) Show that there exists a surjection $\pi_A : E(A) \twoheadrightarrow \Gamma$ such that $ele^{-1} = \pi_A(e)(l)$ for all $e \in E(A)$ and $l \in L$.

   (b) Let A and A' be two simple central finite-dimensional algebras over K, containing L, such that $C_A(L) = L$ and $C_{A'}(L) = L$. Show that A is isomorphic to A' (as a K-algebra) if and only there exists a group isomorphism $f : E(A) \to E(A')$ such that $f(l) = l$ for each $l \in L^\times$ and $\pi_{A'} \circ f = \pi_A$.

3. Let L/K be a finite Galois extension such that $\Gamma = Gal(L/K)$ is cyclic of degree n. Show that $Br(L/K) \cong K^\times/N_{L/K}(L^\times)$, where $N_{L/K} : L^\times \to K^\times$ is the norm map $l \mapsto \prod_{\gamma \in \Gamma} \gamma(l)$.

   Hint: Fix a generator σ of Γ. For every $b \in K^\times$ consider the algebra $S(b)$ generated by the field L and an element x_σ with relations $x_\sigma^n = b$ and $x_\sigma l = \sigma(l)x_\sigma$ for all $l \in L$. Show that $S(b)$ is a central simple algebra over K, and that the map $b \mapsto [S(b)]$ induces an isomorphism $K^\times/N_{L/K}(L^\times) \cong Br(L/K)$.

4. Using the previous question prove the theorem of Wedderburn (that every finite division algebra is commutative) and the theorem of Frobenius (that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite-dimensional division algebras over $\mathbb{R}$).

5. Let A be a simple central algebra over K of dimension n^2.

   (a) Show that there exists a map $\mathrm{Nrd} : A \to K$ such that for every splitting field L of A and every isomorphism $f : A \otimes_K L \cong Mat_n(L)$ of L-algebras, we have $\mathrm{Nrd}(a) = \det(f(a \otimes 1))$ for each $a \in A$. (The map Nrd is called *the reduced norm* map.)

   Hint: Choose a finite Galois extension L/K splitting A, an isomorpism $f : A \otimes_K L \cong Mat_n(L)$ of L-algebras and consider the map $\mathrm{Nrd} : A \to L$ given by $\mathrm{Nrd}(a) = \det(f(a \otimes 1))$. Show that

      i. Nrd does not depend on the choice of f.
      ii. Nrd takes values in K.
      iii. Nrd does not depend on the choice of L.

   (b) For each $a \in A$, let $L_a \in End_K(A)$ be the endomorphism defined by $L_a(x) = ax$. Show that $\det(L_a) = \mathrm{Nrd}(a)^n$.

   (c) Show that A is a division algebra if and only if $\mathrm{Nrd}(a) \neq 0$ for every $a \neq 0$.