Characteristic Classes

S2D4: Seminar on Homotopy Theory

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A characteristic class is a cohomology class associated to a vector bundle, which carries a great deal of information about its structure. The goal of this seminar is to study vector bundles, introduce characteristic classes, and then use these classes to prove interesting statements in algebra, geometry, and topology. The main reference for the seminar will be Characteristic Classes by J.W. Milnor and J.D. Stasheff [MS74]. However, we will use Vector Bundles and K-theory by Hatcher [Hat03] for the construction of the Stiefel-Whitney classes.

Prerequisites: We will assume some familiarity with homology and cohomology, as covered in Topology I. I also strongly recommend taking Topology II concurrently.

Logistics: The preliminary meeting will take place on **Wednesday 15 January** at **2pm**. If you cannot attend the preliminary meeting but think you might like to give a talk, please email me!

Other than the preliminary meeting, the seminar will take place on **Wednesdays 16:15-17:45** during the **Summer 2025** semester.

Please contact me at least **two weeks** prior to the date of your talk, so that we can set up a meeting to discuss your plan for your talk, and discuss any questions you may have about the material.

1. **9 April 2025.** *Vector Bundles I.* Introduce smooth manifolds and the tangent space. Introduce vector bundles, and give some examples. This talk follows Chapters 1 and 2 of [MS74].
2. **16 April 2025.** *Vector Bundles II.* Introduce basic constructions involving vector bundles. This talk follows Chapter 3 of [MS74].
3. **23 April 2025.** *Stiefel-Whitney Classes I.* Introduce Stiefel-Whitney classes, and the axioms that characterize them. Discuss some of their basic properties, and their consequences (for example, the Whitney duality theorem). Prove that $\mathbb{P}^n$ cannot be parallelizable if $n+1$ is a power of 2. This talk follows section 4.1 of [MS74].
4. **30 April 2025.** *Stiefel-Whitney Classes II.* More applications of Stiefel-Whitney classes, including the existence of division algebras, and immersions of projective spaces into $\mathbb{R}^n$. Introduce Stiefel-Whitney numbers, and discuss their applications to cobordism classes of manifolds. This talk follows section 4.2-4.4 of [MS74].
5. **7 May 2025.** *Grassmanian Manifold and Universal Bundle.* Construct the Grassmanian manifolds and the universal bundle. Discuss the classification of vector bundles over a paracompact space by maps to the Grassmanian. Present the isomorphism between the ring of characteristic classes and the cohomology of the Grassmanian. This talk follows chapter 5 of [MS74].
6. **21 May 2025.** *A Cell Structure for Grassmanian Manfiolds.* Give a CW-structure for the infinite Grassmanian $Gr_n(\mathbb{R}^\infty)$. This talk follows chapter 6 of [MS74]. See also Chapter 2.1 of [Hat03].
7. **28 May 2025.** *The Cohomology Ring $H^*(Gr_n; \mathbb{Z}/2)$.* Compute the cohomology of $Gr_n(\mathbb{R}^\infty)$, and prove the uniqueness of Stiefel-Whitney classes. This talk follows chapter 7 of [MS74]. See also Chapter 3.1 of [Hat03].

8. **4 June 2025.** *Existence of Stiefel-Whitney Classes.* Construct the Stiefel-Whitney classes, following Ch. 3.1 of [Hat03].

9. **18 June 2025.** *The Thom Isomorphism Theorem.* Introduce the Thom space, define the Thom class, and prove the Thom isomorphism. This talk follows chapter 10 of [MS74].

10. **25 June 2025.** *Oriented Bundles and the Euler Class.* Introduce orientations and Euler classes. Discuss the relationship between Euler and Stiefel-Whitney classes. This talk follows chapter 9 of [MS74].

11. **2 July 2025.** *Computations in a Smooth Manifold I.* Characteristic classes of the normal bundle and the tangent bundle of a smooth manifold. Discuss the relationship between embeddings of a manifold into $\mathbb{R}^n$ and the characteristic classes of its normal bundle. Discuss the relationship between orientability of a manifold and the orientability of its tangent bundle. This talk follows sections 11.1-11.2 of [MS74].

12. **9 July 2025.** *Computations in a Smooth Manifold II.* Poincare duality and the diagonal class. Relate Euler class to Euler characteristic. This talk follows sections 11.3-11.5 of [MS74].

13. **16 July 2025.** *Obstructions.* Applications of Stiefel-Whitney and Euler classes to obstruction theory. This talk follows chapter 12 of [MS74].

References

[Hat03] Allen Hatcher. Vector bundles and k-theory. *Im Internet unter http://www. math. cornell. edu/˜ hatcher*, 2003.

[MS74] John Willard Milnor and James D Stasheff. *Characteristic classes*. Number 76. Princeton university press, 1974.